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CONTROL
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Disturbance decoupling of multi-input multi-output discrete-time nonlinear systems by static measurement feedback

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Abstract. This paper addresses the disturbance decoupling problem (DDP) for nonlinear systems, extending the results for continuous-time systems into the discrete-time case. Sufficient conditions are given for the solvability of the problem. The notion of the rank of a one-form is used to find the static measurement feedback that solves the DDP whenever possible. Moreover, necessary and sufficient conditions are given for single-input single-output systems, as well as for multi-input multi-output systems under the additional assumption.

Key words: nonlinear systems, discrete-time systems, disturbance decoupling, static measurement feedback.

1. INTRODUCTION

The disturbance decoupling problem (DDP) for a discrete-time nonlinear control system by state feedback has been addressed in many papers (see [2,3,7,8,14,16]). Most papers extend the results known for continuous-time systems into the discrete-time domain (e.g. [6,10,17]), describing the control system by smooth or analytic difference equations. Few studies address the DDP for discrete-time nonlinear control systems using output feedback (e.g. [15,18,20], see also [13]), whereas only [18] treats explicitly the case of static measurement feedback, which is the topic of our paper. However, in [18] necessary and sufficient conditions are given only for single-input single-output (SISO) systems. Papers [15] and [20] focus on dynamic measurement feedback. In [20] the controlled output is a vector function of the measured output, having possibly less components than the measured output itself. Therefore, the above solution may be considered only as a partial solution. Paper [15] provides a full algorithmic solution for the problem using dynamic feedback. In both papers the novel algebraic approach, called the algebra of functions (see [22]), is applied.

Only a few papers address the problem of continuous-time nonlinear control systems ([1,11,19,21]). Paper [19] studies the problem using static measurement feedback, in [11] the feedback considered is restricted to pure dynamic measurement feedback, and papers [1,21] focus on dynamic measurement feedback.

Our goal is to extend the results of [19] for discrete-time nonlinear control systems. Moreover, the results of [19] were given for multi-input single-output (MISO) systems, whereas the present paper addresses the multi-input multi-output (MIMO) case. The preliminary results of this paper (for MISO systems only) were presented at the 18th International Conference on Process Control ([12]).

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2. PRELIMINARIES

Consider a discrete-time nonlinear control system

$$\begin{aligned} x(t+1) &= f(x(t),u(t),w(t)), \\ y(t) &= h(x(t)), \\ z(t) &= k(x(t)), \end{aligned} \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$, the control input $u(t) \in \mathbb{R}^m$, the disturbance input $w(t) \in \mathbb{R}^\nu$, the output to be controlled $y(t) \in \mathbb{R}^p$, and the measured output $z(t) \in \mathbb{R}^\mu$. Assume that f, h, and k are meromorphic functions of their arguments. Throughout the paper we also assume that system (1) is generically submersive, i.e.

$$\operatorname{rank} \frac{\partial f(x(t),u(t),w(t))}{\partial (x(t),u(t),w(t))} = n$$

everywhere except on the set of zero measure.

Under submersivity assumption we can construct the inversive difference field[1] of meromorphic functions in variables $x(t)$, $u(t)$, $w(t)$ and a finite number of their (independent) forward and backward shifts associated with system (1), which we denote as $\mathscr{K}^*$. Note that not all the variables are independent because of the relationships defined by (1) and in the computations the dependent variables have to be expressed via the independent ones. For example, $x(t+1)$ has to be replaced by $f(x(t),u(t),w(t))$. See [9] for the details how to construct $\mathscr{K}^*$.

Define the vector spaces $\mathscr{X} = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t)\}$, $\mathscr{Z} = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}z(t)\}$, $\mathscr{U} = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}u(t+k), k \geq 0\}$, $\mathscr{W} = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}w(t+k), k \geq 0\}$, and $\mathscr{E} = \mathscr{X} + \mathscr{U} + \mathscr{W}$.

Definition 1. ([4])**.** *The relative degree r of the output $y(t)$ is defined by*

$$r := \min\{i \in \mathbb{N} | \mathrm{d}y(t+i) \notin \mathscr{X}\}.$$

If such an integer does not exist, define $r := \infty$.

The static measurement feedback of the form $u(t) = F(z(t),v(t))$ is called regular if F is invertible with respect to $v(t)$, i.e. if there exists an inverse function $\alpha := F^{-1}$ such that $v(t) = \alpha(z(t),u(t))$.

Problem Statement. Given a nonlinear system of the form (1), the goal is to find, if possible, a regular static measurement feedback of the form

$$u(t) = F(z(t),v(t)),$$

such that every controlled output $y_i(t)$, $i = 1,\dots,p$, of the closed-loop system satisfies the following conditions:
(i) $\mathrm{d}y_i(t+k) \in \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t),\mathrm{d}v(t),\dots,\mathrm{d}v(t+k-r_i)\}, \forall k \geq r_i$,
(ii) $\mathrm{d}y_i(t+r_i) \notin \mathscr{X}$,
where r_i is the relative degree of $y_i(t)$ with respect to $u(t)$. Condition (i) represents the independence of the output of the closed-loop system from the disturbance, whereas condition (ii) represents the output controllability of the closed-loop system.

Analogously to the continuous-time case (see [19]), define the subspaces $\Omega_i \subset \mathscr{X}$ for every output $y_i(t)$ $(i = 1,\dots,p)$ by

$$\begin{aligned} \Omega_i &:= \{\omega(t) \in \mathscr{X} | \forall k \in \mathbb{N} : \omega(t+k) \\ &\in \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t),\mathrm{d}y_i(t+r_i),\dots,\mathrm{d}y_i(t+r_i+k-1)\}\}. \end{aligned}$$

[1] An inversive difference field is a pair consisting of a field $\mathscr{K}^*$ and an automorphism σ of $\mathscr{K}^*$, shortly denoted just by $\mathscr{K}^*$. The role of σ is here played by the forward shift operator that takes the variables at the time instant t to the next time instant $t+1$.

The subspaces Ω_i will be important in solving the DDP, because the forward shifts of a one-form $\omega(t) \in \Omega_i$ do not depend explicitly on inputs $u(t)$ and $w(t)$.

To simplify the presentation of the proof of Lemma 1 below, we omit the index i. That is, instead of y_i and Ω_i, $i = 1,\dots,p$, we just write y and Ω, respectively.

Lemma 1. *The subspace Ω may be computed as the limit of the following algorithm*:

$$\begin{aligned} \Omega^0 &= \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t)\}, \\ \Omega^{k+1} &= \{\omega(t) \in \Omega^k | \omega(t+1) \in \Omega^k \\ &\quad +\operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r)\}\}, \qquad k \geq 0. \end{aligned} \tag{2}$$

Proof. We show below that sequence Ω^k converges and in the limit we get Ω. Consider a subspace Ω^k. By (2), $\Omega^{k+1} \subset \Omega^k$ or $\Omega^{k+1} = \Omega^k$. Since the subspace Ω^k is a finite-dimensional vector space, at certain step k^*+1, $\Omega^{k^*} = \Omega^{k^*+1}$. Thus sequence (2) converges and the limit is Ω^{k^*}. We show now that $\Omega = \Omega^{k^*}$. Suppose $\omega(t) \in \Omega^{k^*}$. Then, by (2)

$$\omega(t+1) \in \Omega^{k^*-1} + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r)\}$$

and so $\omega(t+1) = \tilde{\omega}(t) + \xi \mathrm{d}y(t+r)$ for some $\tilde{\omega}(t) \in \Omega^{k^*-1}$ and function $\xi \in \mathscr{K}^*$. Since $\tilde{\omega}(t) \in \Omega^{k^*-1}$, by (2)

$$\tilde{\omega}(t+1) \in \Omega^{k^*-2} + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r)\}$$

and so the forward shift of $\omega(t+1)$ is

$$\omega(t+2) \in \Omega^{k^*-2} + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r), \mathrm{d}y(t+r+1)\}.$$

Continuing in the same way, we get

$$\omega(t+k^*) \in \Omega^0 + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r), \dots, \mathrm{d}y(t+r+k^*-1)\},$$

which means that $\omega(t) \in \Omega$. We showed that if $\omega(t) \in \Omega^{k^*}$, then $\omega(t) \in \Omega$, i.e. $\Omega^{k^*} \subset \Omega$.

Now suppose that $\omega(t) \in \Omega$. Then, by definition of Ω,

$$\omega(t+k^*) \in \mathscr{X} + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r), \dots, \mathrm{d}y(t+r+k^*-1)\}.$$

As $\Omega^0 = \mathscr{X}$,

$$\omega(t+k^*) = \tilde{\omega}(t) + \xi_1 \mathrm{d}y(t+r) + \ldots + \xi_{k^*} \mathrm{d}y(t+r+k^*-1),$$

where $\tilde{\omega}(t) \in \Omega^0$ and $\xi_1, \dots, \xi_{k^*} \in \mathscr{K}^*$. The backward shift $\tilde{\omega}(t-1) \in \Omega^1$, because $\tilde{\omega}(t-1) \in \Omega^0$ and $\tilde{\omega}(t) \in \Omega^0 + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r)\}$. Note that $\mathrm{d}y(t+r-1) \in \Omega^{k^*}$, because $\mathrm{d}y(t+r) \in \Omega^l + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r)\}$ for every $l \geq 0$. Thus the backward shift of $\omega(t+k^*)$ is

$$\omega(t+k^*-1) \in \Omega^1 + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r), \dots, \mathrm{d}y(t+k^*-2)\}.$$

Continuing in the same way, we get

$$\omega(t+1) \in \Omega^{k^*-1} + \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}y(t+r)\}.$$

Thus $\omega(t) \in \Omega^{k^*}$ and we have shown that $\Omega \subset \Omega^{k^*}$. Above we showed that $\Omega^{k^*} \subset \Omega$, so $\Omega = \Omega^{k^*}$. □

Next we will show how Ω changes under the regular static measurement feedback $u(t) = F(z(t), v(t))$. Denote by $\overline{\mathscr{K}^*}$ the field of meromorphic functions in variables $x(t)$, $v(t)$, $w(t)$ and a finite number of their independent forward and backward shifts and define the vector spaces $\overline{\mathscr{X}} = \operatorname{span}_{\overline{\mathscr{K}^*}}\{\mathrm{d}x(t)\}$,

$\overline{\mathscr{V}} = \text{span}_{\overline{\mathscr{K}^*}}\{\mathrm{d}v(t+k), k \geq 0\}$, $\overline{\mathscr{W}} = \text{span}_{\overline{\mathscr{K}^*}}\{\mathrm{d}w(t+k), k \geq 0\}$, $\overline{\mathscr{E}} = \overline{\mathscr{X}} + \overline{\mathscr{V}} + \overline{\mathscr{W}}$. Analogously to [21] we can prove that there exists an isomorphism $\Phi : \mathscr{E} \to \overline{\mathscr{E}}$ such that if Ω_{cl} is the subspace for the closed-loop system, then $\Omega_{\text{cl}} = \Phi(\Omega)$.

Let $\omega(t) \in \mathscr{E}$. In general, $\omega(t)$ is a linear combination over $\mathscr{K}^*$ of a certain number of standard basis elements of $\mathscr{E}$. However, it is often possible to find a linearly independent set of exact one-forms with less elements than those basis elements of $\mathscr{E}$ in terms of which $\omega(t)$ can be expressed. For example, a one-form $\omega(t) = (x_2(t)u_1(t) + u_2(t))\mathrm{d}x_1(t) + x_2(t)x_1(t)\mathrm{d}u_1(t) + x_1(t)\mathrm{d}u_2(t)$, as a linear combination of $\mathrm{d}x_1(t)$, $\mathrm{d}u_1(t)$, and $\mathrm{d}u_2(t)$, can be expressed as a linear combination of two exact one-forms, $\mathrm{d}(x_1(t)u_1(t))$ and $\mathrm{d}(x_1(t)u_2(t))$.

Definition 2. ([5]). *Let γ be the minimal number of linearly independent exact one-forms necessary to express a one-form $\omega(t)$. Then $\omega(t)$ is said to be of rank γ.*

Note that $1 \leq \gamma \leq n$. For example, if the rank γ of a one-form $\omega(t)$ is 1, then $\omega(t) = \xi \mathrm{d}\alpha$ and thus $\omega(t) \wedge \mathrm{d}\omega(t) = 0$. In the general case, if the rank γ is k, then $\omega(t) \wedge (\mathrm{d}\omega(t))^{(k)} = 0$, where $(\mathrm{d}\omega(t))^{(k)} = \mathrm{d}\omega(t) \wedge \ldots \wedge \mathrm{d}\omega(t)$ is k-fold wedge product.

We prove the following lemma for MIMO systems, providing an alternative formulation of the system to be disturbance decoupled. It allows us to check whether the system is disturbance decoupled or not. The lemma will be used later in the proof of the main result of the paper (i.e. Theorem 1).

Lemma 2. *Under the assumption that the relative degrees r_i of the outputs $y_i(t)$ are finite*, *system* (1) *is disturbance decoupled iff*

$$\mathrm{d}y_i(t+r_i) \in \Omega_i + \text{span}_{\mathscr{K}^*}\{\mathrm{d}u(t)\} \tag{3}$$

for $i = 1, \ldots, p$.

Proof.

Necessity. Assume that system (1) is disturbance decoupled, i.e.

$$\mathrm{d}y_i(t+k) \in \text{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t), \ldots, \mathrm{d}u(t+k-r_i)\} \tag{4}$$

for $k \geq r_i$ and

$$\mathrm{d}y_i(t+r_i) \notin \text{span}_{\mathscr{K}^*}\{\mathrm{d}x(t)\}. \tag{5}$$

In particular, $\mathrm{d}y_i(t+r_i) \in \text{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t)\}$. Rewrite the latter as

$$\mathrm{d}y_i(t+r_i) \in \mathscr{X} + \text{span}_{\mathscr{K}^*}\{\mathrm{d}u(t)\}. \tag{6}$$

Thus there exists a one-form $\omega_0(t) \in \mathscr{X}$ and a function $\xi \in \mathscr{K}^*$ such that $\mathrm{d}y_i(t+r_i) = \omega_0(t) + \xi \mathrm{d}u(t)$. We are going to show that $\omega_0(t) \in \Omega_i$. Assume contrarily that $\omega_0(t) \notin \Omega_i$. The forward shift of $\mathrm{d}y_i(t+r_i) \in \text{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t)\}$ is

$$\mathrm{d}y_i(t+r_i+1) \in \text{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}w(t), \mathrm{d}u(t), \mathrm{d}u(t+1)\},$$

which yields a contradiction with (4). Thus, $\omega_0 \in \Omega_i$ and we can rewrite (6) as $\mathrm{d}y_i(t+r_i) \in \Omega_i + \text{span}_{\mathscr{K}^*}\{\mathrm{d}u(t)\}$.

Sufficiency. Assume that condition (3) is fulfilled for system (1). We must show that system (1) satisfies conditions (4) and (5). Because r_i is the relative degree of $y_i(t)$, (5) is satisfied. Because of (3),

$$\mathrm{d}y_i(t+r_i) = \omega_0(t) + \xi \mathrm{d}u(t),$$

where $\omega_0(t) \in \Omega_i$ and $\xi \in \mathscr{K}^*$. Since $\omega_0(t) \in \Omega_i$,

$$\omega_0(t+l) \in \text{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}y_i(t+r_i), \ldots, \mathrm{d}y_i(t+r_i+l-1)\}$$

for all $l \geq 0$. Thus

$$\mathrm{d}y_i(t+r_i+l) \in \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}y_i(t+r_i), \ldots, \mathrm{d}y_i(t+r_i+l-1), \mathrm{d}u(t+l)\}$$

for all $l \geq 0$. Hence

$$\mathrm{d}y_i(t+r_i+l-1) \in \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}y_i(t+r_i), \ldots, \mathrm{d}y_i(t+r_i+l-2), \mathrm{d}u(t+l-1)\}$$

and

$$\mathrm{d}y_i(t+r_i+l) \in \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}y_i(t+r_i), \ldots, \mathrm{d}y_i(t+r_i+l-2), \mathrm{d}u(t+l-1), \mathrm{d}u(t+l)\}.$$

Continuing in the same way, we get

$$\mathrm{d}y_i(t+r_i+l) \in \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x(t), \mathrm{d}u(t), \ldots, \mathrm{d}u(t+l)\}.$$

Replacing l by $l = k - r_i$, we get (4) and thus sufficiency is fulfilled. □

We are going to use the subspaces Ω_i ($i = 1, \ldots, p$) and the concept of the rank of a one-form to give a sufficient condition for the DDP.

3. MAIN RESULTS

The following theorem gives sufficient conditions for solvability of the DDP by static measurement feedback.

Theorem 1. *The DDP for system* (1) *is solvable by static measurement feedback if for* $i = 1, \ldots, p$:
(i) $\mathrm{d}y_i(t+r_i) \in \Omega_i + \mathscr{Z} + \mathscr{U}$,
(ii) *there exists a one-form* $\omega(t) \in \mathscr{Z} + \mathscr{U}$ *such that* $\mathrm{d}y_i(t+r_i) - \omega(t) \in \Omega_i$ *and* $\operatorname{rank} \omega(t) = \gamma \leq m$,
(iii) *for any basis* $\{\mathrm{d}\alpha_1(z(t),u(t)), \ldots, \mathrm{d}\alpha_\gamma(z(t),u(t))\}$ of $\omega(t)$,

$$\operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial \alpha(z(t),u(t))}{\partial u(t)}\right] = \gamma, \tag{7}$$

where $\alpha := [\alpha_1, \ldots, \alpha_\gamma]^T$.

Proof. Assume that condition (i) is fulfilled. Under condition (ii) there exists a one-form $\omega(t)$ such that $\mathrm{d}y_i(t+r_i) - \omega(t) \in \Omega_i$, where

$$\omega(t) = \beta_1 \mathrm{d}\alpha_1(z(t),u(t)) + \ldots + \beta_\gamma \mathrm{d}\alpha_\gamma(z(t),u(t)).$$

When condition (iii) is satisfied, then γ one-forms $\mathrm{d}\alpha_j(z(t),u(t))$, $j = 1, \ldots, \gamma$, are independent with respect to the variable $u(t)$. Define for $j = 1, \ldots, \gamma$

$$v_j(t) = \alpha_j(z(t),u(t)). \tag{8}$$

If $\gamma < m$, by renumbering the inputs $u(t)$, if necessary, complete (8) with

$$v_j(t) = u_j(t), \qquad j = \gamma+1, \ldots, m \tag{9}$$

to get an invertible map. Define a static measurement feedback $u(t) = \alpha^{-1}(z(t), v(t))$ as the solution of (8) and (9). Note that this yields

$$\mathrm{d}y_i(t+r_i) \in \Omega_i \oplus \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}v(t)\}$$

and thus by Lemma 2, system (1) is disturbance decoupled. □

Remark. We point to the fact that condition (ii) is not a direct extension of the respective condition in the single-output case. Even if we can find individual one-forms $\omega_i(t)$ such that $\mathrm{d}y_i(t+r_i) - \omega_i(t) \in \Omega_i$, for all $i = 1,\dots,p$, this does not necessarily mean that we can find a single one-form $\omega(t)$ as given in (ii) of Theorem 1.

In case of SISO systems when $m = 1$, (7) and (ii) of Theorem 1 yield

$$\mathrm{rank}_{\mathscr{K}^*}\left[\frac{\partial\alpha(z(t),u(t))}{\partial u(t)}\right] = \gamma = 1.$$

Thus, condition (iii) of Theorem 1 is satisfied if and only if $\gamma = 1$. For SISO systems we can conclude from Theorem 1 a necessary and sufficient condition.

Corollary 1. *For SISO nonlinear control systems the DDP is solvable by regular static measurement feedback iff*
(i) $\mathrm{d}y(t+r) \in \Omega + \mathscr{Z} + \mathscr{U}$,
(ii) *there exists a one-form* $\omega(t) \in \mathscr{Z} + \mathscr{U}$ *such that* $\mathrm{d}y(t+r) - \omega(t) \in \Omega$ *and* $\mathrm{rank}\ \omega(t) = 1$.

Proof.

Necessity. Assume that system (1) is decoupled by the regular static measurement feedback

$$u(t) = \alpha^{-1}(z(t),v(t)), \qquad v(t) = \alpha(z(t),u(t)). \tag{10}$$

Then by Lemma 2

$$\mathrm{d}y(t+r) \in \Omega + \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}v(t)\}. \tag{11}$$

Combining (11) with (10) implies condition (i). Since $\omega(t) = \xi \mathrm{d}(\alpha(z(t),u(t)))$, $\omega(t) \wedge \mathrm{d}\omega(t) = 0$ and rank $\omega(t) = 1$. Thus condition (ii) is also fulfilled.

Sufficiency. Assume that (i) holds. Then

$$\mathrm{d}y(t+r) \in \Omega \oplus \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}z(t),\mathrm{d}u(t)\}.$$

Since by (ii) the rank of the one-form $\omega(t)$ is 1, define $\omega(t) := \lambda \mathrm{d}v(t)$ and so

$$\mathrm{d}y(t+r) \in \Omega \oplus \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}v(t)\},$$

meaning that the system is decoupled. □

In general there is no necessary and sufficient condition for MIMO systems, but under additional assumptions $\Omega_i \cap \mathscr{Z} = \emptyset$ and $\mathrm{d}y_i(t+r_i) \in \Omega_i \oplus \mathscr{Z} + \mathscr{U}$ we can find a necessary and sufficient condition for MISO systems.

Theorem 2. *Assume that* $\Omega_i \cap \mathscr{Z} = \emptyset$ *and* $\mathrm{d}y_i(t+r_i) \in \Omega_i \oplus \mathscr{Z} + \mathscr{U}$. *The DDP is solvable by regular static measurement feedback iff for* $i = 1,\dots,p$
(i) *there exists a one-form* $\omega(t) \in \mathscr{Z} + \mathscr{U}$ *such that* $\mathrm{d}y_i(t+r_i) - \omega(t) \in \Omega_i$ *and* $\gamma := \mathrm{rank}\ \omega(t) \le m$,
(ii) *for any basis* $\{\mathrm{d}\alpha_1(z(t),u(t)),\dots,\mathrm{d}\alpha_\gamma(z(t),u(t))\}$ of $\omega(t)$,

$$\mathrm{rank}_{\mathscr{K}^*}\left[\frac{\partial\alpha(z(t),u(t))}{\partial u(t)}\right] = \gamma.$$

Proof.

Necessity. Assume that system (1) is disturbance decoupled by the regular static measurement feedback $v(t) = \alpha(z(t),u(t))$. By Lemma 2, $\mathrm{d}y_i^{\mathrm{cl}}(t+r_i) \in \Omega_i^{\mathrm{cl}} + \mathscr{V}$, where $\mathscr{V} = \mathrm{span}_{\overline{\mathscr{K}^*}}\{\mathrm{d}v_1(t),\dots,\mathrm{d}v_m(t)\}$ and $y_i^{\mathrm{cl}}(t)$

is the output of the closed-loop system. Because of the isomorphism $\Phi : \mathscr{E} \to \overline{\mathscr{E}}$ described above and feedback $\alpha(z(t), u(t))$, we can write

$$\mathrm{d}y_i(t + r_i) \in \Omega_i + \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}\alpha(z(t), u(t))\}.$$

Thus, there exist a one-form $\tilde{\omega}(t) \in \Omega_i$ and $\xi \in \mathscr{K}^*$ such that

$$\mathrm{d}y_i(t + r_i) = \tilde{\omega}(t) + \xi \mathrm{d}\alpha(z(t), u(t)).$$

The assumption $\mathrm{d}y_i(t + r_i) \in \Omega_i \oplus \mathscr{Z} + \mathscr{U}$ implies that $\tilde{\omega}(t) \in \Omega_i + \mathscr{Z}$. Rewrite $\tilde{\omega}(t) = \tilde{\omega}_0(t) + \tilde{\omega}_z(t)$ for some $\tilde{\omega}_0(t) \in \Omega_i$ and $\tilde{\omega}_z(t) \in \mathscr{Z}$. As in the proof of Lemma 2, we can show that $\tilde{\omega}_z(t) \in \Omega_i$. Due to the assumption $\Omega_i \cap \mathscr{Z} = 0$, we have $\tilde{\omega}_z(t) = 0$. Then define $\omega(t) = \xi \mathrm{d}\alpha(z(t), u(t))$ and the necessity of condition (i) is fulfilled.

As the rank of a one-form $\omega(t)$ is γ,

$$\omega(t) = \beta_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \beta_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t)),$$

where $\beta_i \in \mathscr{K}^*$, $i = 1, \ldots, \gamma$. Suppose, contrarily to the claim of Theorem 2, that (ii) is not fulfilled. Then there exists a one-form

$$\xi_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \xi_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t)) \in \mathscr{Z}.$$

Assume without loss of generality that $\xi_1 \neq 0$. Then $\omega(t)$ can be decomposed into

$$\omega(t) = \tilde{\omega}_z(t) + \eta_2 \mathrm{d}\alpha_2(z(t), u(t)) + \ldots + \eta_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t)),$$

in which

$$\tilde{\omega}_z(t) = \frac{\beta_1}{\xi_1}(\xi_1 \mathrm{d}\alpha_1(z(t), u(t)) + \ldots + \xi_\gamma \mathrm{d}\alpha_\gamma(z(t), u(t))) \in \mathscr{Z}$$

and

$$\eta_i = \beta_i - \frac{\beta_1}{\xi_1}\xi_i,$$

for $i = 2, \ldots, \gamma$. As shown before, if $\tilde{\omega}_z(t) \in \mathscr{Z}$, then necessarily $\tilde{\omega}_z(t) \in \Omega_i$ and since $\Omega_i \cap \mathscr{Z} = 0$, this yields a contradiction. Thus condition (ii) has to be fulfilled.

Sufficiency. As all conditions of Theorem 1 are satisfied, the sufficiency is fulfilled. □

4. EXAMPLES

In this section we give some examples to illustrate the theory.

Example 1. In the first example we consider the SISO system

$$\begin{aligned}
x_1(t+1) &= x_1(t)x_2(t), \\
x_2(t+1) &= \frac{x_4(t)}{u(t)} + x_3, \\
x_3(t+1) &= \frac{u(t)}{x_4(t)}, \\
x_4(t+1) &= w(t) + x_4(t), \\
y(t) &= x_1(t), \\
z(t) &= x_4(t).
\end{aligned} \tag{12}$$

Note that the relative degree of the output $y(t)$ is 2, because

$$\begin{aligned}
\mathrm{d}y(t+1) &= x_1(t)\mathrm{d}x_2(t)+x_2(t)\mathrm{d}x_1(t) \in \mathscr{X},\\
\mathrm{d}y(t+2) &= \left(\frac{x_4(t)}{u(t)}+x_3(t)\right)\mathrm{d}(x_1(t)x_2(t))+x_1(t)x_2(t)\mathrm{d}x_3(t)+x_1(t)x_2(t)\mathrm{d}\left(\frac{x_4(t)}{u(t)}\right) \notin \mathscr{X}.
\end{aligned}$$

Next we find the vector space Ω, using the algorithm defined by (2). First,

$$\Omega^0 = \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_1(t),\mathrm{d}x_2(t),\mathrm{d}x_3(t),\mathrm{d}x_4(t)\}.$$

From

$$\begin{aligned}
\mathrm{d}x_1(t+1) &= x_1(t)\mathrm{d}x_2(t)+x_2(t)\mathrm{d}x_1(t),\\
\mathrm{d}x_2(t+1) &= \mathrm{d}\left(\frac{x_4(t)}{u(t)}\right)+\mathrm{d}x_3(t),\\
\mathrm{d}x_3(t+1) &= \mathrm{d}\left(\frac{u(t)}{x_4(t)}\right),\\
\mathrm{d}x_4(t+1) &= \mathrm{d}w(t)+\mathrm{d}x_4(t),
\end{aligned}$$

we can conclude that $\Omega^1 = \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_1(t),\mathrm{d}x_2(t),\mathrm{d}x_3(t)\}$. In the next step we get $\Omega^1=\Omega^2=\Omega$. Since $\mathrm{d}z(t)=\mathrm{d}x_4(t)$, condition (i) of Corollary 1 is satisfied, i.e. $\mathrm{d}y(t+2)\in\Omega+\mathscr{Z}+\mathscr{U}$. Our next step is to choose $\omega(t)$ such that $\omega(t)\in\mathscr{Z}+\mathscr{U}$ and $\mathrm{d}y(t+2)-\omega(t)\in\Omega$. We can take $\omega(t):=x_1(t)x_2(t)\mathrm{d}(\frac{z(t)}{u(t)})$. Thus the rank of $\omega(t)$ is 1 and condition (ii) of Corollary 1 is satisfied. The disturbance decoupling feedback may be found as the solution of the equation $v(t)=\frac{z(t)}{u(t)}$ with respect to $u(t)$.

Example 2. Consider the MISO system

$$\begin{aligned}
x_1(t+1) &= x_2(t)+x_3(t)+x_5(t),\\
x_2(t+1) &= \ln(u_1(t)x_4(t)),\\
x_3(t+1) &= u_2(t)x_4(t)x_5(t),\\
x_4(t+1) &= w(t)+x_3(t),\\
x_5(t+1) &= x_5(t),\\
y(t) &= x_1(t),\\
z(t) &= x_4(t).
\end{aligned} \tag{13}$$

The relative degree of output $y(t)$ is 2 and

$$\begin{aligned}
\mathrm{d}y(t+2) &= (1+u_2(t)x_4(t))\mathrm{d}x_5(t)+\mathrm{d}(\ln(u_1(t)x_4(t)))\\
&\quad +x_5(t)\mathrm{d}(u_2(t)x_4(t)).
\end{aligned}$$

We can compute, using Lemma 1, that $\Omega=\Omega^2=\mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_5(t)\}$. As $z(t)=x_4(t)$, condition (i) of Theorem 1 is satisfied. Since now we can choose $\omega(t)$ as $\omega(t)=\mathrm{d}(\ln(u_1(t)z(t)))+x_5(t)\mathrm{d}(u_2(t)z(t))$, $\gamma:=\mathrm{rank}\,\omega(t)=2$ and condition (ii) of Theorem 1 is fulfilled. Condition (7) takes the form

$$\mathrm{rank}\begin{pmatrix} \frac{1}{u_1(t)} & 0 \\ 0 & z(t) \end{pmatrix} = 2$$

and thus condition (iii) of Theorem 1 is satisfied. The regular static measurement feedback can be found by solving equations $v_1(t) = \ln((t)u_1(t)z(t))$ and $v_2(t) = u_2(t)z(t)$ with respect to $u_1(t)$ and $u_2(t)$.

The following example shows that conditions of Theorem 1 are not necessary.

Example 3. Consider the system

$$\begin{aligned} x_1(t+1) &= u_1(t)\sin x_4(t) + x_2(t), \\ x_2(t+1) &= u_2(t)\cos x_4(t) + x_3(t), \\ x_3(t+1) &= x_1(t), \\ x_4(t+1) &= w(t) + x_4(t), \\ y(t) &= x_1(t), \\ z(t) &= x_4(t). \end{aligned} \tag{14}$$

Computing subspace Ω, we get $\Omega = \text{span}_{\mathscr{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_3(t)\}$. It follows that the first condition of Theorem 1 is not satisfied:

$$\mathrm{d}y(t+1) = \mathrm{d}x_2(t) + \mathrm{d}(u_1(t)\sin x_4(t)) \notin \Omega + \mathscr{X} + \mathscr{U}.$$

Still, the following feedback solves the DDP:

$$\begin{aligned} u_1(t) &= \frac{v_1(t)}{\sin z(t)}, \\ u_2(t) &= \frac{v_2(t)}{\cos z(t)}. \end{aligned}$$

Example 4. Consider the MIMO system

$$\begin{aligned} x_1(t+1) &= x_2(t) + x_3(t)u_1(t)x_4(t) + u_2(t)x_4(t), \\ x_2(t+1) &= x_2(t) + x_3(t)u_1(t)x_4(t) + u_2(t)x_4(t) + x_3(t), \\ x_3(t+1) &= x_1(t), \\ x_4(t+1) &= w(t) + x_1(t), \\ y_1(t) &= x_1(t), \\ y_2(t) &= x_2(t), \\ y_3(t) &= x_2(t) - x_1(t), \\ z(t) &= x_4(t). \end{aligned} \tag{15}$$

Note that the relative degree of the outputs $y_1(t)$ and $y_2(t)$ is 1 and the relative degree of $y_3(t)$ is 3. As

$$\begin{aligned} \mathrm{d}y_1(t+1) &= \mathrm{d}y_3(t+3) = \mathrm{d}x_2(t) + u_1(t)x_4(t)\mathrm{d}x_3(t) \\ &\quad + x_3(t)\mathrm{d}(u_1(t)x_4(t)) + \mathrm{d}(u_2(t)x_4(t)), \\ \mathrm{d}y_2(t+1) &= \mathrm{d}x_2(t) + (1 + u_1(t)x_4(t))\mathrm{d}x_3(t) \\ &\quad + x_3(t)\mathrm{d}(u_1(t)x_4(t)) + \mathrm{d}(u_2(t)x_4(t)), \end{aligned}$$

subspaces Ω_i are

$$\Omega_1 = \Omega_2 = \Omega_3 = \text{span}_{\mathscr{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t), \mathrm{d}x_3(t)\}.$$

Thus the first condition of Theorem 1 is satisfied. Now, we must choose $\omega(t)$ such that

$$\mathrm{d}y_i(t+r_i) - \omega(t) \in \Omega_i$$

for every output $y_i(t)$. If we choose $\omega(t)$ to be

$$\omega(t) = x_3(t)\mathrm{d}(u_1(t)z(t)) + \mathrm{d}(u_2(t)z(t)),$$

the previous conditions are satisfied and the rank of $\omega(t)$ is 2. Thus the second condition is also satisfied. Finally, the third condition of Theorem 1 is satisfied because

$$\mathrm{rank}\begin{pmatrix} z(t) & 0 \\ 0 & z(t) \end{pmatrix} = 2.$$

The feedback that solves the DDP is

$$\begin{aligned} u_1(t) &= \frac{v_1(t)}{z(t)}, \\ u_2(t) &= \frac{v_2(t)}{z(t)}. \end{aligned}$$

Recall that all results of this paper hold under submersivity assumption. The following example demonstrates that wrong conclusions can be reached if the system is not submersive, but nevertheless Theorem 1 is blindly used to check the solvability of the DDP by regular static measurement feedback.

Example 5. Consider the system

$$\begin{aligned} x_1(t+1) &= x_2(t), \\ x_2(t+1) &= x_3(t) + x_4(t)u_1(t)u_2(t) - x_1(t)x_5(t), \\ x_3(t+1) &= x_2(t)x_4(t), \\ x_4(t+1) &= w(t), \\ x_5(t+1) &= x_4(t), \\ y(t) &= x_1(t), \\ z(t) &= x_4(t). \end{aligned} \tag{16}$$

System (16) is not submersive. If we forget about this fact and check the conditions of Theorem 1, we will get that the DDP is not solvable for system (16) by static measurement feedback.

Really, the relative degree of output $y(t)$ is 2 and the subspace $\Omega = \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t)\}$. Since

$$\mathrm{d}y(t+2) \quad = \quad \mathrm{d}x_3(t) - \mathrm{d}(x_1(t)x_5(t)) + \mathrm{d}(z(t)u_1(t)u_2(t)),$$

condition (i) of Theorem 1 is not satisfied. However, since the system is not submersive, the forward shift of $x_3(t) - x_1(t)x_5(t)$ is 0 and because of that, the following static measurement feedback solves the DDP:

$$\begin{aligned} u_1(t) &= \frac{v_1(t)}{z(t)v_2(t)}, \\ u_2(t) &= v_2(t). \end{aligned}$$

5. CONCLUSION

In this paper the notion of the rank of a one-form and the subspace Ω of differential one-forms was used to solve the DDP for nonlinear discrete-time control systems by static measurement feedback. Sufficient conditions for solvability of the DDP were found for MIMO systems. Necessary and sufficient conditions were derived from the above conditions for SISO systems and for MIMO systems under the additional assumption. These sufficient conditions also provided a procedure to find the static measurement feedback to solve the DDP. As these conditions are very restrictive, further research is necessary. The next step is to extend the results of [21] by addressing dynamic measurement feedback in the framework of differential forms for discrete-time systems. The results can then be compared with those by [15], which were obtained using the tools of the algebra of functions. In addition to the above theoretical problems, the functions in the Mathematica program have been developed for solving the DDP and integrated into the symbolic software package NLControl, developed in the Institute of Cybernetics at Tallinn University of Technology.

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Häiringu kompenseerimine mittelineaarsetest diskreetsetest mitme sisendi ja mitme väljundiga juhtimissüsteemidest staatilise tagasisidega

Arvo Kaldmäe ja Ülle Kotta

On käsitletud mittelineaarsetest juhtimissüsteemidest häiringu kompenseerimist. Tulemused on saadud, üldistades pideva ajaga süsteemide kohta teadaolevad tulemused diskreetsetele süsteemidele. Töös on esitatud piisavad tingimused probleemi lahenemiseks. Kasutades üksvormi astaku definitsiooni, on leitud otsitav staatiline tagasiside, kui see leidub. Lisaks on esitatud tarvilikud ja piisavad tingimused ühe sisendi ning ühe väljundiga süsteemide jaoks ja lisatingimusi rahuldavatele mitme sisendi ning väljundiga süsteemidele.