



Proceedings of the Estonian Academy of Sciences,  
2012, **61**, 2, 113–127

doi: 10.3176/proc.2012.2.04

Available online at [www.eap.ee/proceedings](http://www.eap.ee/proceedings)

THEORETICAL  
PHYSICS

## Generalized Langevin equation with multiplicative trichotomous noise

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Received 21 June 2011, revised 30 October 2011, accepted 12 December 2011, available online 21 May 2012

**Abstract.** The influence of noise flatness and memory-time on the dynamics of a generalized Langevin system driven by an internal Mittag-Leffler noise and by a multiplicative trichotomous noise is studied. In the asymptotic limit at a short memory time the dynamics corresponds to a system with a pure power-law memory kernel for a viscoelastic type friction. However, at long and intermediate memory times the behaviour of the system has a qualitative difference. In particular, a critical memory time and a critical memory exponent have been found, which mark dynamical transitions in the resonant behaviour of the system. The obtained results show that the model considered is quite robust and may be of interest also in cell biology.

**Key words:** generalized Langevin equation, trichotomous noise, Mittag-Leffler noise, stochastic resonance, memory time, memory exponent.

### 1. INTRODUCTION

In recent years increasing attention has been paid to the constructive role of noise in nature – the influence of noise is not restricted to destructive and thermodynamic effects but can have unexpected ordered outcomes. In complex systems an ensemble of conditions far from thermal equilibrium and the influence of environmental fluctuations may give rise to phenomena that are ruled out by the second law of thermodynamics under equilibrium conditions [1]. The examples include stochastic resonance [1,2], noise-induced multistability [3,4], hypersensitive response [5], noise-enhanced stability [6,7], and the ratchet effect [8–10], to name a few. Particularly, the study of anomalous diffusion in complex or disordered media has made substantial progress during the last years [11–19]. For example, the diffusion of mRNAs and ribosomes in the cytoplasm of living cells is anomalously slow [20], and large proteins behave similarly [21]. Even intrinsic conformational dynamics of protein macromolecules can be subdiffusive [16,22]. There are several approaches to describe anomalous diffusion processes, where the dynamical origin of a phenomenon is considered as nonlocality, either in space or time [18]. In the case of nonlocality in time anomalous diffusion is often connected with one of the two prominent underlying stochastic processes, namely, continuous-time random walks [23] and fractional Brownian motion [15]. It should be noted that in some cases both of the above-mentioned underlying stochastic mechanisms are relevant in different time scales. For example, paper [24] has reported experimental evidence to the effect that at short times the motion of lipid granules in living cells are best described by continuous-time random walk subdiffusion, but at longer times the stochastic mechanism is closest to subdiffusive fractional Brownian motion.

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One of the possibilities for modelling anomalous diffusion in physical and biological systems can be formulated in the framework of the generalized Langevin equation (GLE) [15,17–19]. In most cases a GLE is obtained by replacing the usual friction term by a generalized friction term with a power-law memory [15,16,19]. Physically such a friction term has, due to the fluctuation-dissipation theorem, its origin in a non-Ohmic thermal bath, whose influence on the dynamical system is described with a power-law correlated additive noise in the GLE, e.g., with fractional Gaussian noise which is closely related to fractional Brownian motion [15,17]. Although a GLE with a power-law friction kernel is very useful for modelling anomalous diffusion processes, the corresponding power-law correlated noises have some nonphysical properties, e.g., absence of a characteristic memory time and infinite variance. Thus, recently Viñales and Despósito [25] have introduced a more general noise with a Mittag-Leffler correlation function (called the Mittag-Leffler noise) in the GLE. Notably, for certain values of the parameters that characterize this noise one can reproduce a power-law correlation function, a standard Ornstein–Uhlenbeck noise with an exponential one, and a white noise. The behaviour of the GLE with an additive noise has been investigated in some detail [17,19], but it seems that analysis of the potential consequences of interplay between a multiplicative noise and memory effects is still rather rare in literature [26,27]. This is quite unjustified in view of the fact that the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g., living cells, has been well recognized [21,28].

Thus motivated, we have recently considered a GLE with a Mittag-Leffler memory kernel subjected to an external periodic force [29]. The influence of the fluctuating environment was modelled by a multiplicative trichotomous noise and an additive Mittag-Leffler noise. In the long-time limit this model enables an exact solution for the first moment of the output signal and shows that stochastic resonance (SR) is manifested in the dependence of the response of the GLE upon the amplitude of the trichotomous noise. Moreover, the results of [29] predict that the output signal of the GLE depends crucially on the memory time of the Mittag-Leffler noise.

The purpose of the present paper is to provide a comprehensive view of our approach to the GLE with multiplicative trichotomous noise, more fully describing the results published in [29], and expanding upon them. Specifically, amongst other things we will consider comprehensive construction of exact formulas for the first moment of the output of the basic model-system (in [29] these formulas have only been outlined without proofs), and discuss some novel phenomena where the role of memory time and parameters of the multiplicative noise are crucial. We are reporting here the following novel results:

- (i) We will show that at high values of noise flatness the output signal of the GLE exhibits a hypersensitive response to noise amplitude. Particularly, we will demonstrate that the effect is very pronounced at high values of the characteristic memory time of the Mittag-Leffler noise.
- (ii) We will also establish a SR vs the switching rate of the multiplicative noise and show that this effect can be enhanced by variations in the characteristic memory time of the internal noise.

The structure of the paper is as follows. In Section 2 we present the basic model investigated. Exact formulas for the mean particle displacement are derived in Section 3. In Section 4 we analyse the behaviour of the output response and expose the main results of this paper. Section 5 contains some brief concluding remarks.

## 2. MODEL

We start from the traditional GLE model in one selected direction for a particle of the unit mass ( $m = 1$ ) in the fluctuating harmonic potential

$$V(X, t) = (\omega^2 + Z(t)) \frac{X^2}{2}, \quad (1)$$

subjected to a linear friction with a memory kernel  $\eta(t)$ , an additive periodic force, and an internal random force  $\xi(t)$  of zero mean:

$$\ddot{X} + \int_0^t \eta(t-t') \dot{X}(t') dt' + \frac{\partial}{\partial X} V(X, t) = A_0 \sin(\Omega t) + \xi(t), \quad (2)$$

where  $\dot{X}(t) \equiv \frac{dX}{dt}$ ,  $X(t)$  is the particle displacement, and  $A_0$  and  $\Omega$  are the amplitude and the frequency of the harmonic driving force, respectively. The random force  $\xi(t)$  is Gaussian and fully characterized by its autocorrelation function satisfying the fluctuation-dissipation relation

$$\langle \xi(t)\xi(t') \rangle = k_B T \eta(|t-t'|), \quad (3)$$

which in turn is a consequence of the fluctuation-dissipation theorem [30]. In Eq. (3)  $T$  is the absolute temperature of the heat bath and  $k_B$  is the Boltzmann constant. It is well known that if the correlation function (3) is a Dirac delta function, the stochastic process  $X(t)$  described by Eq. (2) with  $Z(t) = A_0 = 0$  is Markovian and its dynamics can be straightforwardly obtained [31]. However, in order to describe the non-Markovian dynamics of an anomalously diffusing particle, one must take into account the memory effects by a long-time tail noise. Usually a power-law correlation function is employed to model such processes [15,16,19]. As in [25], in this paper we assume a more general correlation function modelled as

$$\eta(t) = \frac{\gamma}{\tau^\alpha} E_\alpha \left[ - \left( \frac{|t|}{\tau} \right)^\alpha \right], \quad (4)$$

where  $\tau$  acts as a characteristic memory time,  $\gamma$  is a constant (called friction constant), and the exponent  $\alpha$  can be taken as  $0 < \alpha < 2$ , which is determined by the dynamical mechanism of the physical process considered. The  $E_\alpha(y)$  function denotes the Mittag-Leffler function [32], which behaves as a stretched exponential for short times and as inverse power-law in the long-time regime. Note that if  $\alpha = 1$ , the correlation function (3) with Eq. (4) reduces to an exponential form which describes a standard Orstein-Uhlenbeck process [31]. In the limit  $\tau \rightarrow 0$  the proposed correlation function reproduces a power-law correlation function

$$\langle \xi(t)\xi(t') \rangle \sim \frac{\gamma k_B T}{\Gamma(1-\alpha)(t-t')^\alpha}, \quad (5)$$

which has been previously used to model viscoelastic properties of a medium [13,19]. Moreover, taking the limit  $\alpha \rightarrow 1$  in Eq. (5), we obtain that the noise  $\xi(t)$  corresponds to a white noise and consequently to non-retarded friction. Note that the model (2) with the kernel (4) for free particle, i.e.,  $V = A_0 = 0$ , has been analysed in [25], where it is shown that, for long times  $t \gg \gamma 1/(\alpha - 2) > \tau$ , the particle motion is subdiffusive for  $0 < \alpha < 1$  and superdiffusive for  $1 < \alpha < 2$ :

$$\sigma^2(t) \equiv \langle X^2(t) \rangle - \langle X(t) \rangle^2 \sim \frac{2k_B T}{\gamma \Gamma(1+\alpha)} t^\alpha, \quad t \rightarrow \infty. \quad (6)$$

Let us note that a GLE (2) without either multiplicative noise or periodic force,  $Z = A_0 = 0$ , was considered in [33]. Particularly, at the deterministic initial conditions  $X(0) = x_0$  and  $\dot{X}(0) = v_0$ , explicit expressions of the position mean value  $\langle X(t) \rangle$  and the variance  $\sigma^2(t)$  are obtained in the case of an overdamped limit (i.e., discarding the inertial term  $\ddot{X}$  in Eq. (2)):

$$\langle X(t) \rangle = v_0 G(t) + x_0 (1 - \omega^2 I(t)), \quad (7)$$

$$\sigma^2(t) = k_B T [2I(t) - G^2(t) - \omega^2 I^2(t)], \quad (8)$$

where

$$I(t) = \frac{1}{\omega^2} \left[ 1 - \rho E_\alpha \left( -\rho \frac{\omega^2}{\gamma} t^\alpha \right) \right], \quad (9)$$

$$G(t) = \frac{d}{dt} I(t), \quad (10)$$

$$\rho = \frac{\gamma}{\gamma + \omega^2 \tau^\alpha}. \quad (11)$$

The asymptotic behaviour of the moments  $\langle X(t) \rangle$  and  $\sigma^2(t)$  in the long-time limit can be obtained by using the asymptotic behaviour of the Mittag-Leffler functions [32]. Then, for  $\omega^2 \tau^\alpha \gg \gamma + \omega^2 \tau^\alpha$  the asymptotic expression for  $\langle X \rangle$  and  $\sigma^2$  is given by

$$\langle X(t) \rangle \approx \frac{\gamma \sin(\alpha\pi)}{\omega^2 \pi} \left( \frac{x_0 \Gamma(\alpha)}{t^\alpha} + \frac{v_0 \Gamma(\alpha+1)}{\omega^2 t^{\alpha+1}} \right), \quad (12)$$

$$\sigma^2(t) \approx \frac{k_B T}{\omega^2} \left( 1 - \frac{\gamma^2 \sin^2(\alpha\pi) \Gamma^2(\alpha)}{\omega^4 \pi^2 t^{2\alpha}} \right). \quad (13)$$

Note that the asymptotic expressions (12) and (13) have the same form as obtained in the case of the pure power-law memory kernel [33]. Moreover, as opposed to the free particle diffusion, the variance of the displacement approaches its equilibrium value due to the confining potential.

Fluctuations of the eigenfrequency  $\omega$  (see Eq. (1)) are expressed as a trichotomous process  $Z(t)$  [34]. Although both dichotomous and trichotomous noises may be useful in modelling natural coloured fluctuations, the latter is more flexible, including all cases of dichotomous noise [34,35]. Furthermore, it is remarkable that for trichotomous noises the flatness parameter  $\kappa$  can be anything from 1 to  $\infty$ , unlike the flatness for Gaussian coloured noise,  $\kappa = 3$ , and symmetric dichotomous noise,  $\kappa = 1$ . This extra degree of freedom can prove useful in modelling actual fluctuations. The trichotomous process is a random stationary Markovian process that consists of jumps between three values  $a$ ,  $0$ , and  $-a$ . The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$p_s(a) = p_s(-a) = q, p_s(0) = 1 - 2q, \quad (14)$$

with  $0 < q \leq 1/2$ . The mean value of  $Z(t)$  and the correlation function are

$$\langle Z(t) \rangle = 0, \quad \langle Z(t+\tau)Z(t) \rangle = 2qa^2 e^{-v\tau}. \quad (15)$$

It can be seen that the switching rate  $v$  is the reciprocal of the noise correlation time  $\tau_c$ , i.e.  $\tau_c = 1/v$ . The flatness parameter  $\kappa$  of the noise  $Z(t)$  proves to be a very simple expression of the probability  $q$

$$\kappa := \frac{\langle Z^4(t) \rangle}{\langle Z^2(t) \rangle^2} = \frac{1}{2q}. \quad (16)$$

The probabilities  $W_n(t)$  that  $Z(t)$  is in the state  $n \in \{1, 2, 3\}$ ,  $z_1 = a$ ,  $z_2 = 0$ ,  $z_3 = -a$ , at the time  $t$  evolve according to the master equation

$$\frac{d}{dt} W_n(t) = v \sum_{m=1}^3 S_{nm} W_m(t), \quad (17)$$

where

$$S_{nm} = \begin{pmatrix} q-1 & q & q \\ 1-2q & -2q & 1-2q \\ q & q & q-1 \end{pmatrix}. \quad (18)$$

The transition probabilities  $T_{ij} = p(z_i, t + \tau | z_j, t)$  between the states  $z_n$ ,  $n = 1, 2, 3$ , can be represented by means of the transition matrix  $T_{ij}$  of the trichotomous process as follows:

$$T_{ij} = \delta_{ij} + (1 - e^{-v\tau}) S_{ij}, \quad (19)$$

where  $\delta_{ij}$  is the Kronecker symbol. The trichotomous process is a particular case of the Kangaroo process [36]. It is remarkable that the results of the present paper can be interpreted in terms of cross-correlation intensity between two dichotomous noises. Namely, the trichotomous noise  $Z(t)$  can be

represented as the sum of two cross-correlated zero-mean symmetric dichotomous noises  $Z_1(t)$  and  $Z_2(t)$ , i.e.,

$$Z(t) = Z_1(t) + Z_2(t).$$

The dichotomous noises  $Z_1(t)$  and  $Z_2(t)$  are characterized as follows:  $z_1, z_2 \in \{(1/2)a, -(1/2)a\}$  with  $v_1 = v_2 = v$  and the correlation function

$$\langle Z_i(t)Z_j(t') \rangle = \rho_{ij} \frac{a^2}{4} e^{-v|t-t'|}, \quad i, j = 1, 2, \quad (20)$$

where  $\rho_{ii} = 1$  and  $\rho_{ij} = \rho \in (-1, 1)$  with  $i \neq j$  being the cross-correlation intensity of the noises  $Z_1(t)$  and  $Z_2(t)$ . In this case the probability  $q = (1 + \rho)/4$ , whence it follows that the correlation coefficient  $\rho$  and the flatness  $\kappa$  of the trichotomous noise  $Z(t)$  must be related as

$$\kappa = \frac{2}{1 + \rho}. \quad (21)$$

It is obvious that the noise flatness  $\kappa = 2$  corresponds to  $\rho = 0$ , i.e., to the case of two statistically independent dichotomous noises. Let us note that such a cross-correlation between dichotomous noises may result from either of the two following reasons: the two noises are either partly of the same origin or are influenced by the same factors. Notably, earlier some cross-correlation-induced effects have been considered in the context of ratchet models in [37,38], where it has also been suggested that cross-correlation between coloured noises may provide some understanding as to why structurally very similar motor proteins with two heads, such as kinesin and dynein motor families, move in opposite directions on the micro-tubules despite sharing the same environment and experiencing the same periodicity, like with the conventional kinesin and ncd [39].

### 3. EXACT SOLUTION

In what follows we will analyse the behaviour of the first moment  $\langle X \rangle$  of the output of model (2) in the subdiffusive case, i.e.,  $0 < \alpha < 1$ . To find the first moment of  $X$ , we use the well-known Shapiro–Loginov procedure [40], which for a trichotomous noise  $Z(t)$  yields

$$\frac{d}{dt} \langle Z\Phi \rangle = \left\langle Z \frac{d}{dt} \Phi \right\rangle - v \langle Z\Phi \rangle, \quad (22)$$

where  $\Phi$  is an arbitrary functional of the process  $Z(t)$ . From Eqs (1), (2), and (22), we thus obtain an exact linear system of six first-order integro-differential equations for six variables,  $x_1 = \langle X \rangle$ ,  $x_2 = \langle \dot{X} \rangle$ ,  $x_3 = \langle ZX \rangle$ ,  $x_4 = \langle Z\dot{X} \rangle$ ,  $x_5 = \langle Z^2 X \rangle$ ,  $x_6 = \langle Z^2 \dot{X} \rangle$ :

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega^2 x_1 - x_3 - \int_0^t \eta(t-t') x_2(t') dt' + A_0 \sin(\Omega t), \\ \dot{x}_3 &= -v x_3 + x_4, \\ \dot{x}_4 &= -v x_4 - \omega^2 x_3 - x_5 - e^{-vt} \int_0^t \eta(t-t') e^{vt'} x_4(t') dt', \\ \dot{x}_5 &= -v x_5 + x_6 + 2qa^2 v x_1, \\ \dot{x}_6 - 2qa^2 \dot{x}_2 &= -v (x_6 - 2qa^2 x_2) - a^2 (1 - 2q) x_3 - \omega^2 (x_5 - 2qa^2 x_1) \\ &\quad - e^{-vt} \int_0^t \eta(t-t') \left[ e^{vt'} (x_6(t') - 2qa^2 x_2(t')) \right] dt'. \end{aligned} \quad (23)$$

The solution of Eqs (23) can be formally represented in the form

$$x_i(t) = \sum_{k=1}^6 H_{ik}(t)x_k(0) + A_0 \int_0^t [H_{i2}(t') + 2qa^2 H_{i6}(t')] \sin [\Omega(t-t')] dt', \quad (24)$$

where the constants of integration  $x_k(0)$  are determined by initial conditions. The relaxation functions  $H_{ik}(t)$  with the initial conditions  $H_{ik}(0) = \delta_{ik}$  can be obtained by means of the Laplace transformation technique. From Eqs (23) we obtain the following system of algebraic linear equations for  $\widehat{H}_{ik}(s)$ :

$$\begin{aligned} s\widehat{H}_{1k} - \widehat{H}_{2k} &= \delta_{1k}, \\ (s + \widehat{\eta}(s))\widehat{H}_{2k} + \omega^2\widehat{H}_{1k} + \widehat{H}_{3k} &= \delta_{2k}, \\ (s + \nu)\widehat{H}_{3k} - \widehat{H}_{4k} &= \delta_{3k}, \\ [s + \nu + \widehat{\eta}(s + \nu)]\widehat{H}_{4k} + \omega^2\widehat{H}_{3k} + \widehat{H}_{5k} &= \delta_{4k}, \\ (s + \nu)\widehat{H}_{5k} - \widehat{H}_{6k} - 2qa^2\nu\widehat{H}_{1k} &= \delta_{5k}, \\ [s + \nu + \widehat{\eta}(s + \nu)](\widehat{H}_{6k} - 2qa^2\widehat{H}_{2k}) + (1 - 2q)a^2\widehat{H}_{3k} + \omega^2(\widehat{H}_{5k} - 2qa^2\widehat{H}_{1k}) &= \delta_{6k} - 2qa^2\delta_{2k}, \end{aligned} \quad (25)$$

where  $k = 1, \dots, 6$  and  $\widehat{H}_{ik}(s)$  is the Laplace transform of  $H_{ik}(t)$ , i.e.,

$$\widehat{H}_{ik}(s) = \int_0^{\infty} e^{-st} H_{ik}(t) dt.$$

The solution of Eqs (25) for  $\widehat{H}_{1k}(s)$  reads as

$$\begin{aligned} \widehat{H}_{11}(s) &= \frac{1}{D(s)} \left\{ -2qa^2[s + \nu + \widehat{\eta}(s + \nu)] + (s + \widehat{\eta}(s)) \right. \\ &\quad \left. \times \left[ [(s + \nu)(s + \nu + \widehat{\eta}(s + \nu)) + \omega^2]^2 - (1 - 2q)a^2 \right] \right\}, \\ \widehat{H}_{12}(s) &= \frac{1}{D(s)} \left\{ [(s + \nu)(s + \nu + \widehat{\eta}(s + \nu)) + \omega^2]^2 - a^2 \right\}, \\ \widehat{H}_{13}(s) &= -\frac{1}{D(s)} [s + \nu + \widehat{\eta}(s + \nu)] \times [(s + \nu)(s + \nu + \widehat{\eta}(s + \nu)) + \omega^2], \\ \widehat{H}_{14}(s) &= -\frac{1}{D(s)} [(s + \nu)(s + \nu + \widehat{\eta}(s + \nu)) + \omega^2], \\ \widehat{H}_{15}(s) &= \frac{1}{D(s)} [s + \nu + \widehat{\eta}(s + \nu)], \\ \widehat{H}_{16}(s) &= \frac{1}{D(s)}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} D(s) &= (1 - 2q)a^2 [(s + \nu)\widehat{\eta}(s + \nu) - s\widehat{\eta}(s) + \nu(2s + \nu)] \\ &\quad + [(s + \nu)(s + \nu + \widehat{\eta}(s + \nu)) + \omega^2] \\ &\quad \times \left\{ [s(s + \widehat{\eta}(s)) + \omega^2] [(s + \nu)(s + \nu + \widehat{\eta}(s + \nu)) + \omega^2] - a^2 \right\}, \end{aligned} \quad (27)$$

$$\widehat{\eta}(s) = \frac{\gamma s^{\alpha-1}}{1 + (\tau s)^\alpha}, \quad (28)$$

and

$$\widehat{h}(s) = \widehat{H}_{12}(s) + 2qa^2 \widehat{H}_{16}(s) = \frac{1}{D(s)} \left\{ [(s+v)(s+v+\widehat{\eta}(s+v)) + \omega^2]^2 - (1-2q)a^2 \right\}. \quad (29)$$

One can check the stability of solution (24), which, according to the results of paper [41], means that the solutions  $s_j$  of the equation  $D(s) = 0$  cannot have roots with a positive real part. This requirement is met if the inequality

$$a^2 < a_{\text{cr}}^2 = \omega^2 \left[ \omega^2 + v^2 + \frac{\gamma v^\alpha}{1 + (\tau v)^\alpha} \right]^2 \left[ \omega^2 + 2q \left( \frac{\gamma v^\alpha}{1 + (\tau v)^\alpha} + v^2 \right) \right]^{-1} \quad (30)$$

holds. Henceforth in this work we shall assume that condition (30) is fulfilled. Thus in the long-time limit,  $t \rightarrow \infty$ , the memory about the initial conditions will vanish as

$$\sum_{k=1}^6 H_{1k}(t)x_k(0) = \frac{\gamma \widehat{h}(0)x_1(0)}{\Gamma(1-\alpha)t^\alpha} + O(t^{-(1+\alpha)}) \quad (31)$$

and the average particle displacement  $\langle X \rangle_{as} \equiv \langle X \rangle|_{t \rightarrow \infty}$  is given by

$$\langle X \rangle_{as} = A_0 \int_0^t h(t-t') \sin(\Omega t') dt'. \quad (32)$$

From Eq. (32) it follows that the complex susceptibility  $\chi(\Omega)$  of the dynamical system (2) is given by

$$\chi(\Omega) = \chi'(\Omega) + i\chi''(\Omega) = \widehat{h}(-i\Omega), \quad (33)$$

where  $\chi'(\Omega)$  and  $\chi''(\Omega)$  are the real and the imaginary parts of the susceptibility, respectively. Equation (32) can be written by means of the complex susceptibility as

$$\langle X \rangle_{as} = A \sin(\Omega t + \Theta), \quad (34)$$

with the output amplitude

$$A = A_0 \cdot |\chi| \quad (35)$$

and the phase shift

$$\Theta = \arctan \left( -\frac{\chi''}{\chi'} \right). \quad (36)$$

Using Eqs (27) and (29), we obtain for  $A$  that

$$A^2 = A_0^2 \frac{C_1}{C_2}, \quad (37)$$

where

$$\begin{aligned} C_1 &= [g_1^2 + g_3^2 - (1-2q)a^2]^2 + 4(1-2q)a^2 g_3^2, \\ C_2 &= [g_2^2 + g_4^2] C_1 + 4qa^2 \{ (g_1^2 + g_3^2) (g_3 g_4 - g_1 g_2) + a^2 [q(g_1^2 + g_3^2) + (1-2q)(g_1 g_2 + g_3 g_4)] \} \end{aligned} \quad (38)$$

and

$$\begin{aligned}
 g_1 &= \omega^2 + \nu^2 - \Omega^2 + \frac{\gamma(\nu^2 + \Omega^2)^{\frac{\alpha}{2}} \left[ \cos(\alpha\varphi) + \tau^\alpha(\nu^2 + \Omega^2)^{\frac{\alpha}{2}} \right]}{1 + \tau^{2\alpha}(\nu^2 + \Omega^2)^\alpha + 2(\nu^2 + \Omega^2)^{\frac{\alpha}{2}} \tau^\alpha \cos(\alpha\varphi)}, \\
 g_2 &= \omega^2 - \Omega^2 + \frac{\gamma\Omega^\alpha \left[ \cos\left(\frac{\pi\alpha}{2}\right) + (\tau\Omega)^\alpha \right]}{1 + (\tau\Omega)^{2\alpha} + 2(\tau\Omega)^\alpha \cos\left(\frac{\pi\alpha}{2}\right)}, \\
 g_3 &= 2\Omega\nu + \frac{\gamma(\nu^2 + \Omega^2)^{\frac{\alpha}{2}} \sin(\alpha\varphi)}{1 + \tau^{2\alpha}(\nu^2 + \Omega^2)^\alpha + 2(\nu^2 + \Omega^2)^{\frac{\alpha}{2}} \tau^\alpha \cos(\alpha\varphi)}, \\
 g_4 &= \frac{\gamma\Omega^\alpha \sin\left(\frac{\pi\alpha}{2}\right)}{1 + (\tau\Omega)^{2\alpha} + 2(\tau\Omega)^\alpha \cos\left(\frac{\pi\alpha}{2}\right)}, \\
 \varphi &= \arctan\left(\frac{\Omega}{\nu}\right).
 \end{aligned} \tag{39}$$

Finally, from Eqs (27), (29), and (33) one can conclude that the real and the imaginary parts of the susceptibility are given by

$$\begin{aligned}
 \chi' &= \frac{1}{C_2} \left[ g_2 C_1 - 2qa^2 g_1 (g_1^2 + g_3^2 - (1 - 2q)a^2) \right], \\
 \chi'' &= \frac{1}{C_2} \left[ g_4 C_1 + 2qa^2 g_3 (g_1^2 + g_3^2 + (1 - 2q)a^2) \right].
 \end{aligned}$$

The analytical expressions (37)–(39) belong to the main results of this work. They fully determine the behaviour of the average oscillator displacement in response to system parameters in the long-time limit.

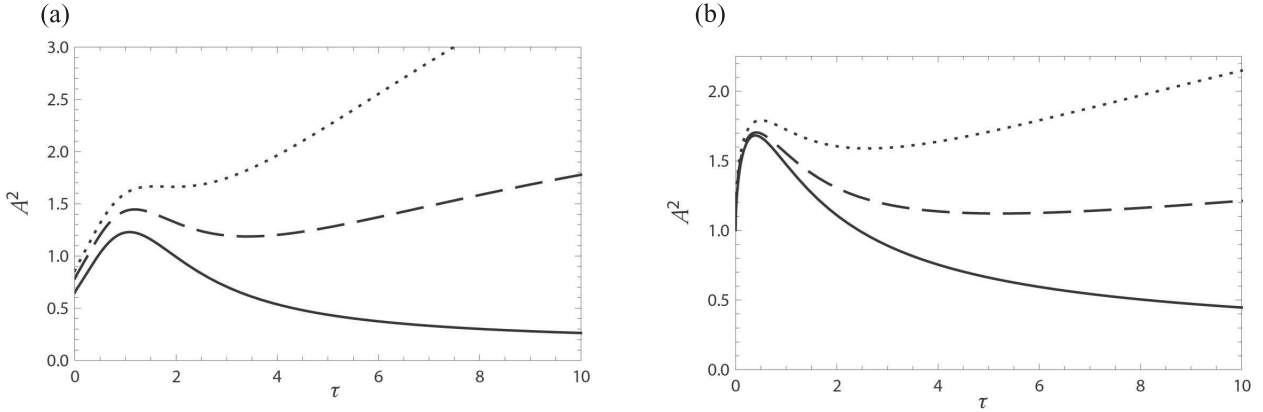
#### 4. TRICHOTOMOUS-NOISE-INDUCED RESONANCE

By the use of Eqs (37)–(39) we can now explicitly obtain the behaviour of  $A(\tau)$  for any combination of the system parameters  $\alpha$ ,  $\gamma$ ,  $a$ ,  $\Omega$ ,  $q$ , and  $\omega$ . Figure 1 depicts the behaviour of the response  $A$  versus the characteristic memory time  $\tau$  for different values of the noise flatness parameter  $\kappa$  and the memory exponent  $\alpha$ . In this figure one observes resonance versus  $\tau$ , which apparently gets more and more pronounced as the flatness parameter  $\kappa = 1/2q$  increases. Thus, as a rule, there exists an optimal memory time at which the response of the output signal to the external periodic force has a maximal value.

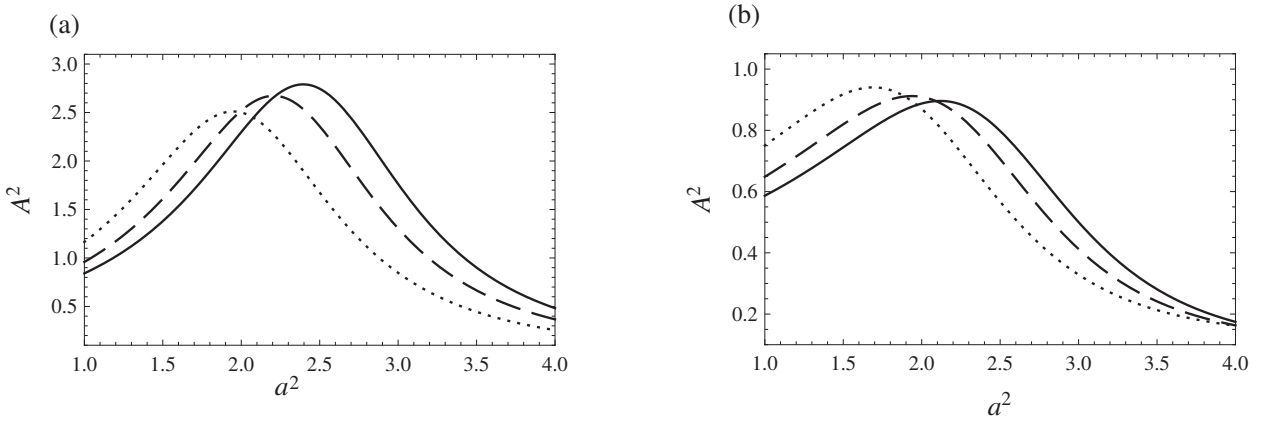
Our next task is to examine the dependence of the response  $A$  on the noise amplitude  $a$ . In Fig. 2 we depict the behaviour of  $A(a)$  for various values of the system parameters  $\tau$  and  $q$ . As shown in Fig. 2, all curves exhibit a resonance-like maximum at some values of  $a$ , i.e., a typical SR phenomenon appears with increase in  $a$ . Next we consider, in brief, another interesting SR phenomenon – hypersensitive response to noise amplitude. A peculiarity of Fig. 3 is the rapid decrease in  $A^2$  from maximum to minimum as  $a$  increases. It is noteworthy that in the case of dichotomous noise such an effect is absent. The effect is very pronounced at low values of the switching rate  $\nu$  and at large values of the flatness parameter  $\kappa = 1/2q$ . To throw some light on the above-mentioned effect, we shall now briefly consider the behaviour of the SR characteristic  $A^2$  in the parameter regimes:

$$\nu^2 \ll \frac{\gamma}{\tau^\alpha} \ll q|\omega^2 - \Omega^2| \ll \omega^2, \quad q \ll 1, \quad \tau \gg \frac{1}{\Omega}, \tag{40}$$

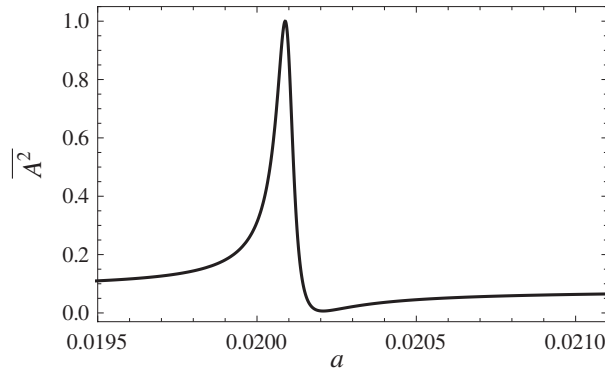




**Fig. 1.** A plot of the dependence of the response function  $A$  on the characteristic time  $\tau$  at  $A_0 = \omega = \Omega = 1$ ,  $\gamma = 0.8$ ,  $a^2 = 0.5$ ,  $\nu = 0.1$ . Solid line:  $q = 0.5$ ; dashed line:  $q = 0.4$ ; dotted line:  $q = 0.35$ . Panel (a):  $\alpha = 0.8$ ; panel (b):  $\alpha = 0.5$ .



**Fig. 2.** Stochastic resonance for the response function  $A$  vs the multiplicative noise amplitude  $a$  at  $A_0 = \omega = 1$ ,  $\Omega = 1.8$ ,  $\nu = \alpha = 0.1$ ,  $\gamma = 1.4$ . Solid line:  $\tau = 5.0$ ; dashed line:  $\tau = 1.0$ ; dotted line:  $\tau = 0.1$ . Panel (a):  $q = 0.4$ ; panel (b):  $q = 0.2$ .



**Fig. 3.** A plot of the dependence of the response function  $A$  on the noise amplitude  $a$  in a region of hypersensitive response (Eqs (37) and (40)). System parameter values:  $A_0 = \omega = 1$ ,  $\Omega = 0.99$ ,  $\nu = 10^{-6}$ ,  $\gamma = 2 \times 10^{-4}$ ,  $\tau = 10$ ,  $\alpha = 0.1$ , and  $q = 5 \times 10^{-3}$ . The value of  $A^2$  at the local maximum is  $A_m^2 = 30292$ ;  $\bar{A}^2 \equiv A^2/A_m^2$ .

and

$$v^2 \ll \gamma \Omega^\alpha \ll q |\omega^2 - \Omega^2| \ll \omega^2, \quad q \ll 1, \quad \tau \ll \frac{1}{\Omega}. \quad (41)$$

In these cases it follows from Eqs (37)–(39) that  $A^2$  reaches the maximum

$$A_{\max}^2 \approx A_0^2 \frac{\tau^{4\alpha} \Omega^{2\alpha} q^2}{\gamma^2 \sin^2\left(\frac{\pi\alpha}{2}\right)}, \quad \tau \gg \frac{1}{\Omega}, \quad (42)$$

$$A_{\max}^2 \approx A_0^2 \frac{q^2}{\gamma^2 \Omega^{2\alpha} \sin^2\left(\frac{\pi\alpha}{2}\right)}, \quad \tau \ll \frac{1}{\Omega}, \quad (43)$$

at

$$a = a_{\max} \approx |\omega^2 - \Omega^2|, \quad (44)$$

and the minimum

$$A_{\min}^2 \approx A_0^2 \frac{\gamma^2 \sin^2\left(\frac{\pi\alpha}{2}\right)}{\tau^{4\alpha} \Omega^{2\alpha} q^2 (\omega^2 - \Omega^2)^4}, \quad \tau \gg \frac{1}{\Omega}, \quad (45)$$

$$A_{\min}^2 \approx A_0^2 \frac{\gamma^2 \Omega^{2\alpha} \sin^2\left(\frac{\pi\alpha}{2}\right)}{q^2 (\omega^2 - \Omega^2)^4}, \quad \tau \ll \frac{1}{\Omega}, \quad (46)$$

at

$$a = a_{\min} \approx \frac{|\Omega^2 - \omega^2|}{\sqrt{1-2q}}. \quad (47)$$

For sufficiently strong inequalities (40) and (41),  $A_{\min}^2$  tends to zero and  $A_{\max}^2$  grows up to very large values. Thus, in the cases considered the response  $A$  is extremely sensitive to a small variation in  $a$ :  $\Delta a = a_{\min} - a_{\max} \approx q |\Omega^2 - \omega^2| \ll \omega^2$ . It is important to note that such a phenomenon gets more pronounced as the characteristic memory time  $\tau$  increases (see Eqs (42) and (45)).

The existence of a SR vs  $a$  effect depends strongly on other system parameters. From Eqs (37)–(39) one can easily find the necessary and sufficient conditions for the emergence of SR due to noise amplitude variations. Namely, a nonmonotonic behaviour of  $A(a)$  appears in the stability region,  $0 < a < a_{\text{cr}}$  (see Eq. (30)), for the parameter regime where the following inequalities hold:

$$a_{\text{cr}}^2 > a_m^2 > 0, \quad (48)$$

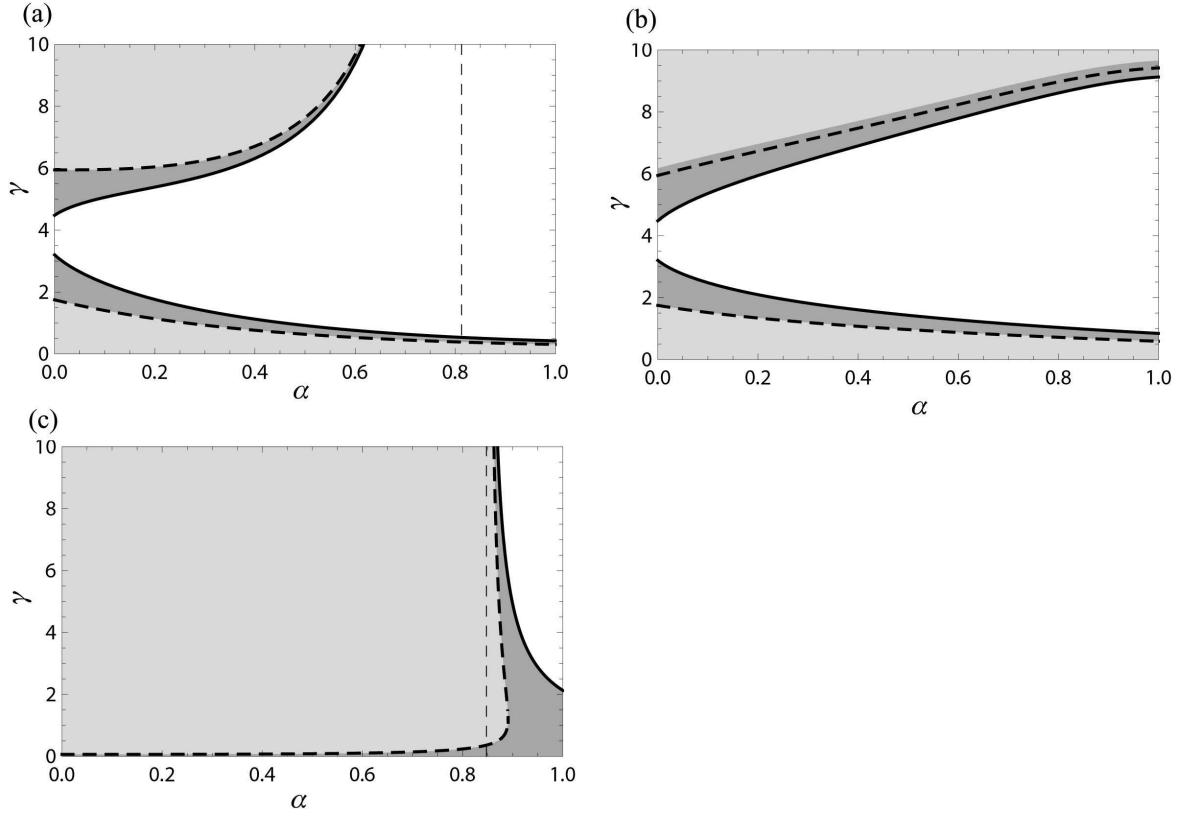
where  $a_m^2$  is the positive solutions of the equation

$$x^2(1-2q) \left\{ (1-2q)(g_1^2 + g_3^2)(g_3g_4 - g_1g_2) + 2(g_1^2 - g_3^2) [q(g_1^2 + g_3^2) + (1-2q)(g_1g_2 + g_3g_4)] \right\} \\ - 2x(g_1^2 + g_3^2)^2 [q(g_1^2 + g_3^2) + (1-2q)(g_1g_2 + g_3g_4)] + (g_1^2 + g_3^2)^3 (g_1g_2 - g_3g_4) = 0. \quad (49)$$

In Figs 4 and 5 conditions (48) are illustrated in the parameter space  $(\gamma, \alpha)$  with three panels in either. The dark grey shaded domains in the figures correspond to those regions of the parameters  $\gamma$  and  $\alpha$  where SR versus  $a$  is possible. Note that in the light grey regions the response  $A(a)$  formally also exhibits a resonance-like maximum, but in those regions the first moment  $\langle X(t) \rangle$  is unstable at the resonance regime, and that renders formula (37) physically meaningless. Two findings can be pointed out. First, if the memory time  $\tau$  is sufficiently small,

$$\tau < \tau_{\text{cr}} = \frac{2}{v + \sqrt{5v^2 + 4\Omega^2}}, \quad (50)$$

there exists a critical memory exponent  $\alpha_{\text{cr}}$ , which marks a sharp transition in the behaviour of the system dynamics. At  $\alpha_{\text{cr}}$ , one of the boundaries  $\gamma_{1,2}(\alpha)$  between the resonance and non-resonance regions tends to



**Fig. 4.** A plot of the phase diagrams for stochastic resonance (SR) in the  $\gamma$ - $\alpha$  plane at  $A_0 = \omega = 1$ ,  $\nu = 0.8$ , and  $q = 0.45$ . In the unshaded region resonance of  $A$  vs the multiplicative noise amplitude  $a$  is impossible. In the light grey region the function  $A(a)$  exhibits a maximum at  $a_m > a_{cr}$ , i.e., at  $a_m$  the first moment of the particle displacement  $\langle X(t) \rangle$  is unstable, see Eq. (30). In the dark grey domain (the stability region) a SR of  $A$  vs  $a$  occurs. The thin dashed line depicts the position of the critical memory exponent  $\alpha_{cr}$ . Panel (a):  $\Omega = 1.8$ ,  $\tau = 0.2$ ; panel (b):  $\Omega = 1.8$ ,  $\tau = 0.85$ ; panel (c):  $\Omega = 0.6$ ,  $\tau = 0.2$ .

infinity. From Eqs (39) and (49) it follows that the critical memory exponent  $\alpha_{cr}$  is determined as a solution of the following equation:

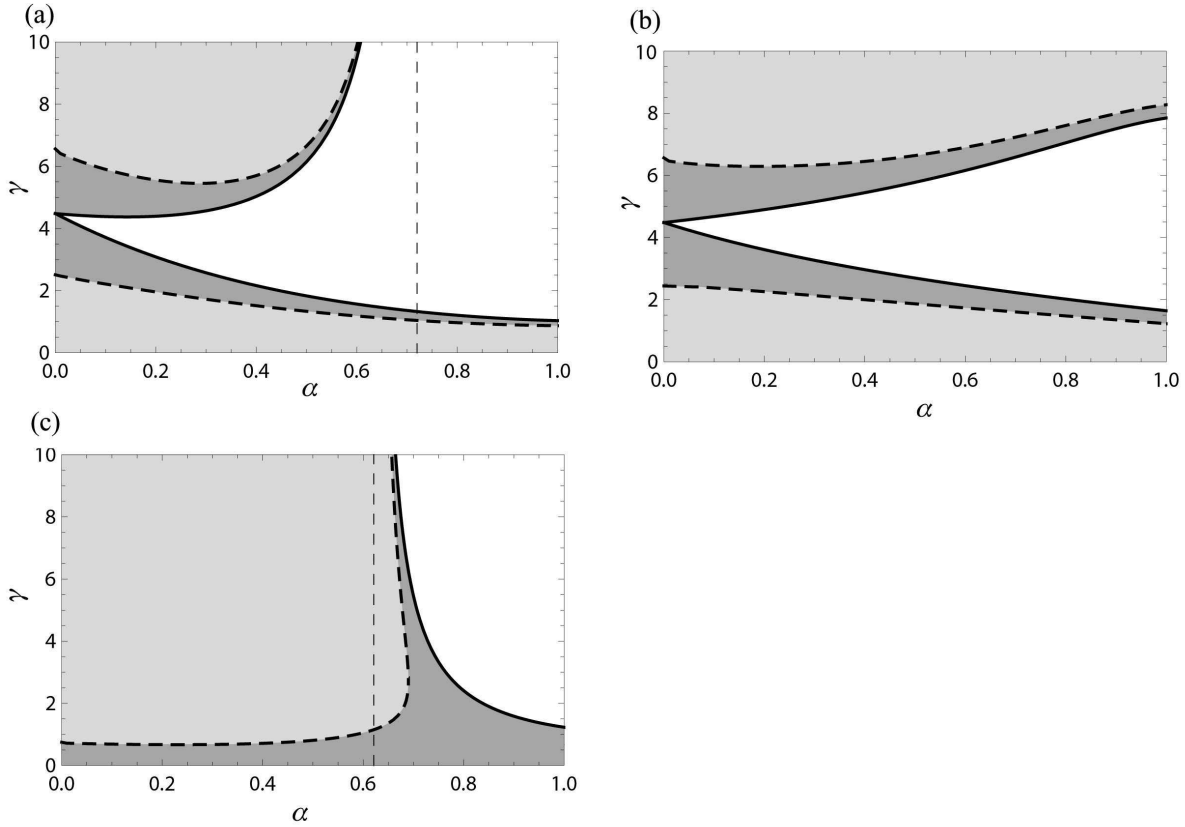
$$\left[ \tau^2 \Omega (\nu^2 + \Omega^2)^{\frac{1}{2}} \right]^{\alpha_{cr}} + \left[ \tau (\nu^2 + \Omega^2)^{\frac{1}{2}} \right]^{\alpha_{cr}} \cos\left(\frac{\alpha_{cr} \pi}{2}\right) + (\tau \Omega)^{\alpha_{cr}} \cos(\alpha_{cr} \varphi) + \cos\left[\alpha_{cr} \left(\varphi + \frac{\pi}{2}\right)\right] = 0. \quad (51)$$

It can be seen from Eq. (51) that the minimal value of a critical memory exponent,  $\alpha_{cr} \geq \alpha_{cr \min}$ , corresponds to the case of a vanishing memory time,  $\tau \rightarrow 0$ , i.e. (see also [26]),

$$\alpha_{cr \min} = \frac{\pi}{\pi + 2 \arctan\left(\frac{\Omega}{\nu}\right)}. \quad (52)$$

Particularly, in the limit  $\tau \rightarrow \tau_{cr}$  the critical exponent  $\alpha_{cr}$  tends to 1. Here we emphasize that in the case of  $\tau > \tau_{cr}$  such a transition of the system dynamics is absent (see Figs 4b and 5b).

The second finding is that depending on the driving frequency  $\Omega$ , three different cases can be discerned where the inequality (50) holds.

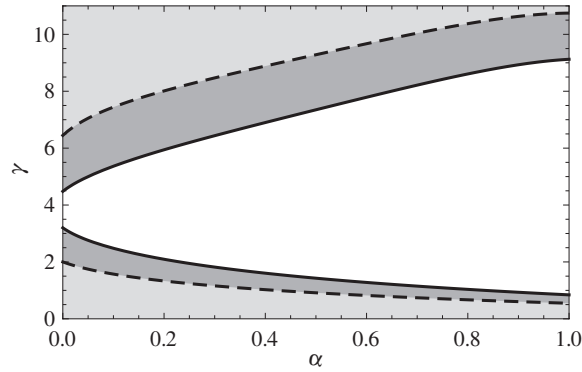


**Fig. 5.** The phase diagram for stochastic resonance vs  $a$  in the  $\gamma$ - $\alpha$  plane in the case of adiabatic noise,  $\nu = 0$ . Panel (a):  $\Omega = 1.8$ ,  $\tau = 0.2$ ; panel (b):  $\Omega = 1.8$ ,  $\tau = 0.85$ ; panel (c):  $\Omega = 0.6$ ,  $\tau = 0.2$ . Other system parameters are the same as in Fig. 4.

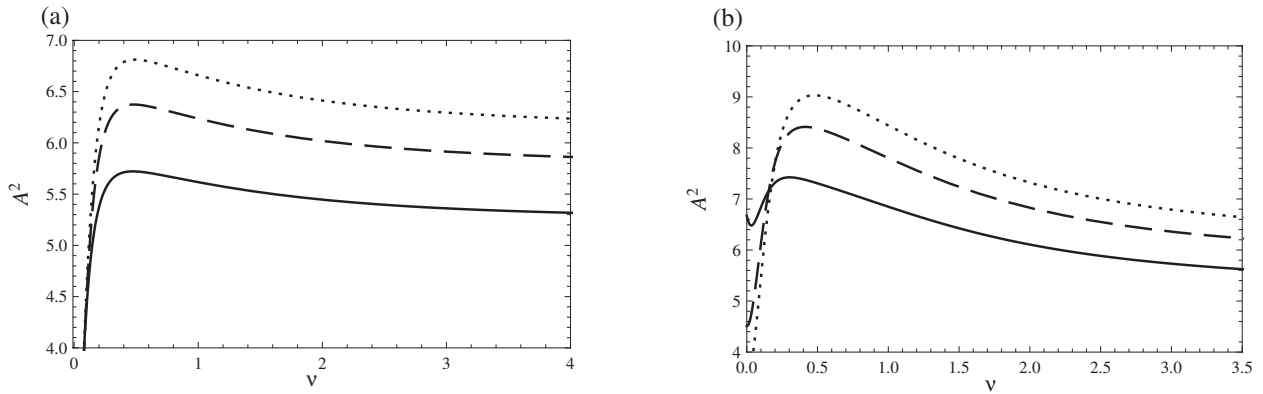
- (i) For  $\Omega^2 < \omega^2$ , SR vs  $a$  appears for all values of  $\gamma$  when  $\alpha < \alpha_{\text{cr}}$ , but if  $\alpha > \alpha_{\text{cr}}$ , there is an upper border  $\gamma(\alpha)$  above which SR is absent (Fig. 4c).
- (ii) In the case of  $\omega^2 < \Omega^2 < \omega^2 + \nu^2$  for  $\alpha < \alpha_{\text{cr}}$  the resonance exists only if  $\gamma > \Omega^2 - \omega^2$ ; in the region  $\alpha > \alpha_{\text{cr}}$  the resonance is absent.
- (iii) At the driving frequency regime  $\Omega^2 > \omega^2 + \nu^2$ , if  $\alpha < \alpha_{\text{cr}}$ , there are two disconnected regions (Fig. 4a) where SR vs  $a$  is possible.

An important observation here is that the region where the resonance is not possible grows as the noise switching rate  $\nu$  increases (cf. Figs 4 and 5). Thus, in this case a variation in the values of the friction parameter  $\gamma$  induces reentrant transitions between different dynamical regimes. Namely, an increase in  $\gamma$  can induce transitions from a regime where SR vs  $a$  is possible to a regime where SR is absent, but SR appears again through a reentrant transition at higher values of  $\gamma$ . Note that in the case of a long memory time,  $\tau > \tau_{\text{cr}}$ , the critical memory exponent  $\alpha_{\text{cr}}$  is absent and the dynamical system (2) behaves qualitatively similarly to the case of  $\alpha < \alpha_{\text{cr}}$  (cf. Fig. 4a, b). It is remarkable that the critical memory exponent  $\alpha_{\text{cr}}$ , the critical memory time  $\tau_{\text{cr}}$ , and the boundaries  $\gamma_{1,2}(\alpha)$  between the resonance and non-resonance regions are independent of the noise flatness  $\kappa$ . Only the stability region depends on  $\kappa$ , increasing as does the noise flatness (cf. Figs 4b and 6).

The phenomenon of SR is not restricted to the nonmonotonic dependence of  $A$  on the noise amplitude  $a$ . Figure 7 depicts the behaviour of the response  $A^2$  versus the noise switching rate  $\nu$  for different representative values of the memory time  $\tau$  and the noise flatness parameter  $q = 1/2\kappa$ . In this figure one observes resonance versus  $\nu$ , which apparently gets more and more pronounced as the memory time  $\tau$  increases.



**Fig. 6.** The phase diagram for stochastic resonance (SR) vs  $a$  in the  $\gamma$ - $\alpha$  plane in the case of large flatness. The parameter values:  $A_0 = \omega = 1$ ,  $\nu = 0.8$ ,  $\tau = 0.85$ ,  $\Omega = 1.8$ , and  $q = 0.05$ . In the unshaded region SR vs  $a$  is impossible. In the light grey region the function  $A(a)$  exhibits a maximum at  $a_m > a_{cr}$ ; see Eq. (30). In the dark grey domain SR vs  $a$  occurs (in the stability region,  $a_m < a_{cr}$ ).



**Fig. 7.** Stochastic resonance for  $A^2$  versus the noise switching rate  $\nu$ , computed from Eq. (37) at various values of the parameters  $q$  and  $\tau$ . Other parameter values:  $A_0 = \omega = 1$ ,  $a^2 = \alpha = 0.3$ ,  $\Omega = 0.8$ , and  $\gamma = 0.09$ . Solid line:  $\tau = 0$ ; dashed line:  $\tau = 0.1$ ; dotted line:  $\tau = 1$ . Panel (a):  $q = 0.1$ ; panel (b):  $q = 0.4$ .

## 5. CONCLUSIONS

We have derived, in the long-time regime, the exact formulas (see Eqs (37)–(39)) for the output response of a system with memory described by a generalized Langevin equation under the impact of an external periodic force. The influence of fluctuations of environmental parameters on the dynamics of the system is modelled by a multiplicative three-level noise and by an internal Mittag-Leffler noise. This study is an extension of our recent short conference paper [29], where we presented the model and some initial results about SR vs the amplitude  $a$  of the multiplicative noise. In the current paper we have given a much more detailed analysis of the same model, focusing on the influence of the characteristic memory time  $\tau$  of the friction-kernel on the trichotomous-noise-induced SR. As one of the new results (in comparison with paper [29]) we have established very sensitive response of the mean particle displacement to small variations in the noise amplitude at high values of the multiplicative noise flatness, i.e., the amplitude  $A$  of the output signal of the GLE displays a quick jump from a very high value to a low one as the noise amplitude  $a$  increases but a little. It is important to note that such a phenomenon was previously reported for a stochastic oscillator without memory in [2]. As another new result we have found a nonmonotonic dependence of the response function on the switching rate  $\nu$  of the multiplicative noise (i.e. SR vs  $\nu$ ). It is remarkable that both effects, i.e., the hypersensitive response  $A$  vs  $a$  and the SR vs  $\nu$ , get more and more pronounced as the characteristic

memory time  $\tau$  increases. The results of the present work and [29] show that the model considered is quite robust and may be of interest also in cell biology, where issues of memory and multiplicative coloured noise can be crucial [15,21,28]. A further detailed study is, however, necessary, especially an investigation of the behaviour of the velocity autocorrelation function, which is an important measure for correct interpretation of experimental results [42]. Some hints about the unexpected influence of multiplicative noise on the behaviour of autocorrelation functions can be found in [43], where a model system (similar to model (2)) with multiplicative white noise was considered.

## ACKNOWLEDGEMENTS

The work was supported by the Estonian Science Foundation under grant No. 7319, by the Ministry of Education and Research of Estonia under grant No. SF0132723s06, by the International Atomic Energy Agency under grant No. 14797, and by the European Union through the European Regional Development Fund (Centre of Excellence “Mesosystems: Theory and Applications”, TK114).

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## Multiplikatiivse kolmetasemelise müra toime üldistatud Langevini võrrandis

Erkki Soika, Romi Mankin ja Jaanis Priimets

On uuritud müra tasasusparameetri ja mälu aja mõju üldistatud Langevini võrrandiga kirjeldatava süsteemi dünaamikale. Üldistatud Langevini võrrandile on rakendatud kaht müra: Mittag-Leffleri müra ja multiplikatiivset kolmetasemelist Markovi protsessi. On leitud kriitiline mälu kestus ja astmenäitaja, mille korral avaldub dünaamilises süsteemis resonantsinähtus. Saadud tulemused näitavad, et uuritud mudelil on potentsiaali rakendusteks eri valdkondades, näiteks rakubioloogias.