



On some operator equations in the space of analytic functions and related questions

Mehmet Gürdal* and Filiz Şöhret

Department of Mathematics, Süleyman Demirel University, 32260 Isparta, Turkey; filiz_sohret@hotmail.com

Received 4 October 2011, accepted 19 January 2012, available online 7 May 2013

Abstract. We investigate extended eigenvalues, extended eigenvectors, and cyclicity problems for some convolution operators. By using the Duhamel product technique, we also estimate the norm of the inner derivation operator Δ_A .

Key words: extended eigenvalue, extended eigenvector, α -Duhamel product, starlike region, Frechet space, inner derivation operator.

1. INTRODUCTION AND BACKGROUND

Let $\mathcal{B}(E)$ be an algebra of all continuous linear operators acting on the topological vector space E . The operator equation

$$AX = XB \tag{1}$$

naturally arises in numerous issues of spectral theory of operators, representation theory, stability theory (Lyapunov's equation), etc. For example, if the set of solutions of Eq. (1) contains a boundedly invertible operator X_0 , then A and B are similar, $B = X_0^{-1}AX_0$, and hence have many common spectral properties. In general case, it is of interest to describe the set of all solutions of Eq. (1).

If $B = \lambda A$, $\lambda \in \mathbb{C}$, then following [1], one refers to λ as an extended eigenvalue of A , and each bounded solution X of the equation

$$AX = \lambda XA,$$

i.e., Eq. (1) with $B = \lambda A$, is called an extended eigenvector of A .

In this paper we investigate the so-called extended eigenvalues and extended eigenvectors and cyclicity problems for some convolution operators acting on the space of analytic functions defined on the starlike domain \mathcal{D} of the complex plane. Our investigation is motivated by the results of Nagnibida's paper [11]. By using the Duhamel product method (see [13]), we also give a lower estimate for the inner derivation operator Δ_A defined in the Banach algebra $\mathcal{B}\left(C_A^{(n)}(\mathbb{D})\right)$ by $\Delta_A(X) := AX - XA$.

The integration operator V on $L^p[0, 1]$ ($1 \leq p < \infty$) is defined by $Vf(x) = \int_0^x f(t) dt$. The set of intertwining operators for the pair $\{V^\beta, \lambda V^\beta\}$ with $\beta > 0$ and $\lambda \in \mathbb{C}$ was studied by Malamud in [3,9,10]. Namely, he showed that there exists a nonzero intertwining operator for the pair $\{V^\beta, \lambda V^\beta\}$ only if $\lambda > 0$. Furthermore, the paper [10] provides a description of the set $\{V^\beta\}'_\lambda$ of all intertwining operators for the pair

* Corresponding author, gurdalmehmet@sdu.edu.tr

$\{V^\beta, \lambda V^\beta\}$ for $\lambda > 0$. For $\beta = 1$, the latter result was reproved by another method by Biswas, Lambert, and Petrovic [1], and Karaev [6]. For more details, see [1,2,4,5,9,10].

Let α be a fixed complex number, let \mathcal{D} be a simply connected region in the complex plane \mathbb{C} that is starlike with respect to the point $z = \alpha$ (i.e., $\lambda z + (1 - \lambda)\alpha \in \mathcal{D}$), and let $\mathcal{A}(\mathcal{D})$ be the space of all single-valued and analytic functions in \mathcal{D} that have a topology of uniform convergence on compact subsets. It is well known that $\mathcal{A}(\mathcal{D})$ is a Frechet space. By \mathcal{I}_α we shall denote the integration operator in the space $A(\mathcal{D})$ defined by the formula

$$(\mathcal{I}_\alpha f)(z) = \int_\alpha^z f(t) dt \quad (\forall f \in \mathcal{A}(\mathcal{D})),$$

where the integration is performed over straight-line segments connecting the points α and z ($z \in \mathcal{A}(\mathcal{D})$).

Recall that for $f, g \in A(\mathcal{D})$ their α -Duhamel product is defined by

$$\begin{aligned} \left(f \underset{\alpha}{\otimes} g\right)(z) &= \frac{d}{dz} \int_\alpha^z f(z + \alpha - t) g(t) dt \\ &= \int_\alpha^z f'(z + \alpha - t) g(t) dt + f(\alpha) g(z), \end{aligned} \quad (2)$$

where the integrals are taken over the segment joining the points α and z ($z \in A(\mathcal{D})$). It is easy to see that the α -Duhamel product satisfies all the axioms of multiplication, $A(\mathcal{D})$ is an algebra with respect to $\underset{\alpha}{\otimes}$ as well,

and the function $f(z) \equiv 1$ is the unit element of the algebra $\left(A(\mathcal{D}), \underset{\alpha}{\otimes}\right)$. The operator $\mathcal{D}_f, \mathcal{D}_f^\alpha g := f \underset{\alpha}{\otimes} g$, is called the α -Duhamel operator on $A(\mathcal{D})$.

2. EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS FOR SOME CONVOLUTION OPERATORS

Let $\mathcal{D} \subset \mathbb{C}$ be a starlike region with respect to the origin. For any fixed nonzero function $f \in \mathcal{A}(\mathcal{D})$, let \mathcal{K}_f be the usual convolution operator acting on the space $\mathcal{A}(\mathcal{D})$ by the formula

$$(\mathcal{K}_f g)(z) = (f * g)(z) := \int_0^z f(z-t) g(t) dt.$$

It follows from the classical Titchmarsh convolution theorem and uniqueness theorem for analytic functions that $\ker(\mathcal{K}_f) = \{0\}$. This means that 0 is not an extended eigenvalue of the operator \mathcal{K}_f , and therefore $\text{ext}(\mathcal{K}_f) \subset \mathbb{C} \setminus \{0\}$ (here $\text{ext}(\mathcal{K}_f)$ denotes the set of all extended eigenvalues of the operator \mathcal{K}_f).

The integration operator \mathcal{I} on $\mathcal{A}(\mathcal{D})$ is defined by $\mathcal{I} f(z) = \int_0^z f(t) dt$. Let $f^{\otimes k}$ denote the \otimes -product (which is clearly $\underset{0}{\otimes}$, that is the usual Duhamel product) of f with itself k times for $k \geq 0$, i.e., $f^{\otimes k} := \underbrace{f \otimes \dots \otimes f}_k$, where $f^{\otimes 0}(z) \equiv 1$. If f is a function in $\mathcal{A}(\mathcal{D})$ such that $\{(\mathcal{I} f)^{\otimes n}\}_{n \geq 0}$ is a complete system in $\mathcal{A}(\mathcal{D})$, we will denote by Λ_f the set of all $\lambda \in \mathbb{C} \setminus \{0\}$ for which the diagonal operator

$$D_{\{\lambda\}}(\mathcal{I} f)^{\otimes n} = \lambda^n (\mathcal{I} f)^{\otimes n}, \quad n \geq 0,$$

is continuous in $\mathcal{A}(\mathcal{D})$.

The following theorem gives necessary and sufficient conditions under which the set Λ_f lies in the set $\text{ext}(\mathcal{K}_f)$. Our result is apparently the first result in the “extended theory” for more general convolution

operators, which is an extension of Karaev's result [7, Theorem 2, (ii)]. The related results for the integration operator are considered in [4,7].

Theorem 1. Let $f \in \mathcal{A}(\mathcal{D})$ be a nonzero function. Suppose that the system $\{(\mathcal{J}f)^{\otimes n}\}_{n \geq 0}$ is complete in $\mathcal{A}(\mathcal{D})$. Let $A \in \mathcal{B}(\mathcal{A}(\mathcal{D}))$ be a nonzero operator and $\lambda \in \Lambda_f$ be any number. Then

$$A\mathcal{K}_f = \lambda\mathcal{K}_fA$$

if and only if there exists $\varphi \in \mathcal{A}(\mathcal{D})$ such that $A = \mathcal{D}_\varphi D_{\{\lambda\}}$.

Proof. By using the usual Duhamel product \otimes , which is defined by

$$(f_1 \otimes f_2)(z) := \frac{d}{dz} \int_0^z f_1(z-t)f_2(t) dt,$$

we have that any function $f_1 \in \mathcal{A}(\mathcal{D})$ defines the continuous operator (Duhamel operator) $\mathcal{D}_{f_1}f_2 := f_1 \otimes f_2$, $f_2 \in \mathcal{A}(\mathcal{D})$. Then we have

$$\begin{aligned} \mathcal{K}_f g &= \mathcal{J} \mathcal{D}_f g = z \otimes (f \otimes g) \\ &= (z \otimes f) \otimes g = \mathcal{D}_{z \otimes f} g = \mathcal{D}_{\mathcal{J}f} g \end{aligned}$$

for all $g \in \mathcal{A}(\mathcal{D})$. Thus $\mathcal{K}_f = \mathcal{A}(\mathcal{D}) \mathcal{J}f$.

Now, let $\lambda \in \Lambda_f$ be any number, and suppose that

$$\lambda \mathcal{K}_f A = A \mathcal{K}_f.$$

Then, obviously

$$\lambda^n \mathcal{K}_f^n A g = A \mathcal{K}_f^n g$$

for all $g \in \mathcal{A}(\mathcal{D})$ and $n \geq 0$. In particular, putting $g = 1$ in the last equality, we have

$$A \mathcal{K}_f^n 1 = \lambda^n \mathcal{K}_f^n A 1$$

for all $n \geq 0$. Since $\mathcal{K}_f = \mathcal{D}_{\mathcal{J}f}$, clearly we have

$$\mathcal{K}_f^n 1 = \mathcal{D}_{\mathcal{J}f}^n 1 = (\mathcal{J}f)^{\otimes n} \otimes 1 = (\mathcal{J}f)^{\otimes n}$$

for all $n \geq 0$. This shows that

$$\begin{aligned} A(\mathcal{J}f)^{\otimes n} &= \lambda^n ((\mathcal{J}f)^{\otimes n} \otimes A1) \\ &= \lambda^n (\mathcal{J}f)^{\otimes n} \otimes A1 = \mathcal{D}_{A1}(\lambda^n (\mathcal{J}f)^{\otimes n}) \\ &= \mathcal{D}_{A1} D_{\{\lambda\}} (\mathcal{J}f)^{\otimes n} \end{aligned}$$

for all $n \geq 0$. Since $\{(\mathcal{J}f)^{\otimes n}\}_{n \geq 0}$ is a complete system of the space $\mathcal{A}(\mathcal{D})$ and $D_{\{\lambda\}}$ is a continuous operator on $\mathcal{A}(\mathcal{D})$, it follows from the last equalities that

$$A g = \mathcal{D}_{A1} D_{\{\lambda\}} g$$

for all $g \in \mathcal{A}(\mathcal{D})$, which means that $A = \mathcal{D}_\varphi D_{\{\lambda\}}$, where $\varphi = A1 \in \mathcal{A}(\mathcal{D})$, as desired.

Conversely, let us show that if A has the form $A = \mathcal{D}_\varphi \mathcal{D}_{\{\lambda\}}$, where $\varphi \in \mathcal{A}(\mathcal{D})$, it satisfies the equation $A\mathcal{K}_f = \lambda\mathcal{K}_f A$. In fact, by considering the formula $A\mathcal{K}_f = \mathcal{D}_{\mathcal{J}_f}$, and commutativity of the product \otimes , we obtain

$$\begin{aligned} \lambda\mathcal{K}_f A(\mathcal{J}f)^{\otimes n} &= \lambda\mathcal{K}_f \mathcal{D}_\varphi \mathcal{D}_{\{\lambda\}}(\mathcal{J}f)^{\otimes n} \\ &= \lambda \mathcal{D}_{\mathcal{J}_f} \mathcal{D}_\varphi (\lambda^n (\mathcal{J}f)^{\otimes n}) = \lambda \mathcal{D}_\varphi \mathcal{D}_{\mathcal{J}_f} (\lambda^n (\mathcal{J}f)^{\otimes n}) \\ &= \lambda \mathcal{D}_\varphi (\mathcal{J}f \otimes \lambda^n (\mathcal{J}f)^{\otimes n}) = \mathcal{D}_\varphi \lambda^{n+1} (\mathcal{J}f \otimes (\mathcal{J}f)^{\otimes n}) \\ &= \mathcal{D}_\varphi \lambda^{n+1} (\mathcal{J}f)^{\otimes n+1} = \mathcal{D}_\varphi \mathcal{D}_{\{\lambda\}} (\mathcal{J}f)^{\otimes n+1} \\ &= A(\mathcal{J}f \otimes (\mathcal{J}f)^{\otimes n}) = A \mathcal{D}_{\mathcal{J}_f} (\mathcal{J}f)^{\otimes n} \\ &= A\mathcal{K}_f (\mathcal{J}f)^{\otimes n} \end{aligned}$$

for all $n \geq 0$. By considering completeness of the system $\{(\mathcal{J}f)^{\otimes n}\}_{n \geq 0}$ in $\mathcal{A}(\mathcal{D})$, from the last equalities we deduce that $A\mathcal{K}_f = \lambda\mathcal{K}_f A$. The theorem is proved. \square

3. CYCLIC VECTORS OF CONVOLUTION OPERATOR $\mathcal{K}_{f,\alpha}$

Let \mathcal{D} be a starlike region in the complex plane \mathbb{C} with respect to $z = \alpha$. Our next result describes all cyclic vectors of some convolution operators of the form

$$(\mathcal{K}_{f,\alpha}g)(z) := \int_\alpha^z f(z+\alpha-t)g(t)dt.$$

Theorem 2. Let $f \in \mathcal{A}(\mathcal{D})$, and assume that $\{(\mathcal{J}_\alpha f)^\alpha{}^{\otimes n}\}_{n \geq 0}$ is a complete system in $\mathcal{A}(\mathcal{D})$. If $g \in \mathcal{A}(\mathcal{D})$, then g is a cyclic vector for the convolution operator $\mathcal{K}_{f,\alpha}$ if and only if $g(\alpha) \neq 0$.

Proof. It follows from the definition of α -Duhamel product \otimes_α that

$$\begin{aligned} \mathcal{K}_{f,\alpha}h &= \mathcal{J}_\alpha \mathcal{D}_{f,\alpha}h = (z-\alpha) \otimes_\alpha \left(f \otimes_\alpha h \right) \\ &= \left((z-\alpha) \otimes_\alpha f \right) \otimes_\alpha h = \mathcal{D}_{(z-\alpha) \otimes_\alpha f} h = \mathcal{D}_{\mathcal{J}_\alpha f} h \end{aligned}$$

for all $h \in \mathcal{A}(\mathcal{D})$, which means that $\mathcal{K}_{f,\alpha} = \mathcal{D}_{\mathcal{J}_\alpha f}$. Then according to the condition of the theorem, we obtain that

$$\begin{aligned} E_g &= \text{span} \{ \mathcal{K}_{f,\alpha}^n g : n \geq 0 \} = \text{span} \{ (\mathcal{D}_{\mathcal{J}_\alpha f})^n g : n \geq 0 \} \\ &= \text{span} \left\{ (\mathcal{J}_\alpha f)^\alpha{}^{\otimes n} \otimes_\alpha g : n \geq 0 \right\} = \text{span} \left\{ \mathcal{D}_{g,\alpha} (\mathcal{J}_\alpha f)^\alpha{}^{\otimes n} : n \geq 0 \right\} \\ &= \text{clos}_{\mathcal{D}_{g,\alpha}} \text{span} \left\{ (\mathcal{J}_\alpha f)^\alpha{}^{\otimes k} : k \geq 0 \right\} = \text{clos}_{\mathcal{D}_{g,\alpha}} \mathcal{A}(\mathcal{D}), \end{aligned}$$

so

$$E_g = \text{clos}_{\mathcal{D}_{g,\alpha}} \mathcal{A}(\mathcal{D}).$$

Now, if $g(\alpha) \neq 0$, then by virtue of Nagnibida's result operator $\mathcal{D}_{g,\alpha}$ is invertible in $\mathcal{A}(\mathcal{D})$, which implies that

$$\mathcal{D}_g \mathcal{A}(\mathcal{D}) = \mathcal{A}(\mathcal{D}).$$

Hence $E_g = \mathcal{A}(\mathcal{D})$, which shows that g is a cyclic vector for the convolution operator $\mathcal{K}_{f,\alpha}$.

Conversely, suppose that $g \in \mathcal{A}(\mathcal{D})$ is a cyclic vector for the operator $\mathcal{H}_{f,\alpha}$, that is $E_g = \mathcal{A}(\mathcal{D})$. If $g(\alpha) \neq 0$, it is easy to see from the equality $E_g = \text{clos } \mathcal{D}_{g,\alpha} \mathcal{A}(\mathcal{D})$ that $E_g \subset \{h \in \mathcal{A}(\mathcal{D}) : h(\alpha) = 0\}$, which is impossible because $E_g = \mathcal{A}(\mathcal{D})$. Consequently, $g(\alpha) \neq 0$, which proves the theorem. \square

Since $\{(z - \alpha)^n\}_{n \geq 0}$ is a complete system in $\mathcal{A}(\mathcal{D})$, the next corollary immediately follows from Theorem 2.

Corollary 1. Let \mathcal{J}_α be an integration operator defined on $\mathcal{A}(\mathcal{D})$ by $(\mathcal{J}_\alpha g)(z) = \int_\alpha^z g(t) dt$. Then

$$\text{Cyc}(\mathcal{J}_\alpha) = \{g \in \mathcal{A}(\mathcal{D}) : g(\alpha) \neq 0\},$$

where $\text{Cyc}(\mathcal{J}_\alpha)$ denotes the set of all cyclic vectors of \mathcal{J}_α .

For the related results see [6–8] and Tkachenko [12]; in [7] the analogous results are considered by Karaev for the Banach space $C_A^{(n)}(\mathbb{D})$.

4. ON THE NORM OF INNER DERIVATION OPERATOR

Let A be a fixed linear bounded operator acting on the Banach space $C_A^{(n)}(\mathbb{D})$, which is the space of all n -times continuously differentiable functions on $\overline{\mathbb{D}}$ that are holomorphic on the unit disc \mathbb{D} . In [7], Karaev proved that $C_A^{(n)}(\mathbb{D})$ is an algebra with multiplication of the Duhamel product

$$(f \otimes g)(z) = \frac{d}{dz} \int_0^z f(z-t)g(t) dt. \tag{3}$$

Thus, the Duhamel operator \mathcal{D}_f defined on $C_A^{(n)}(\mathbb{D})$ by $\mathcal{D}_f g := f \otimes g$ is bounded and $\|\mathcal{D}_f\| \leq \|f\|$. On the other hand, it is clear from (3) that $f = f \otimes 1$, and therefore $\|\mathcal{D}_f\| = \|f\|$. In this section, by using this formula we will estimate the norm of the inner derivation operator Δ_A defined on the Banach algebra $\mathcal{B}(C_A^{(n)}(\mathbb{D}))$ by the formula

$$\Delta_A(X) := AX - XA.$$

Obviously

$$\|\Delta_A\| \leq 2\|A\|. \tag{4}$$

The following theorem gives some lower estimate for $\|\Delta_A\|$ in terms of A .

Theorem 3. Let $A \in \mathcal{B}(C_A^{(n)}(\mathbb{D}))$ be a fixed operator. Suppose that for every $X \in \mathcal{B}(C_A^{(n)}(\mathbb{D}))$ there exists a nonzero function $f := f_X \in C_A^{(n)}(\mathbb{D})$ such that

$$((AX - XA)f)(0) \neq 0.$$

Then there exists a constant $C_A > 0$ such that

$$C_A \leq \|\Delta_A\| \leq 2\|A\|.$$

Proof. According to (4), there remains only to prove the left inequality. Indeed, let us denote

$$(AX - XA)f(z) := g(z). \tag{5}$$

Clearly, $g = g_{A,X}$. Since $g(0) \neq 0$, by the result of paper [8, Theorem 1], the Duhamel operator \mathcal{D}_g is invertible in $C_A^{(n)}(\mathbb{D})$. Therefore, there exists a unique $G \in C_A^{(n)}(\mathbb{D})$ such that $G \otimes g = g \otimes G = 1$. Hence, $f \otimes G \otimes g = f$. Thus, it follows from (5) that

$$\mathcal{D}_F (AX - XA) f = f, \quad (6)$$

where $F := f \otimes G$. Clearly, $F = F_{A,X}$. The equality (6) shows that $1 \in \sigma_p(\mathcal{D}_F (AX - XA))$, that is, 1 is the eigenvalue of the operator $\mathcal{D}_F \Delta_A (X)$. Therefore,

$$\begin{aligned} 1 &\leq r(\mathcal{D}_F (AX - XA)) \leq \|\mathcal{D}_F (AX - XA)\| \\ &\leq \|\mathcal{D}_F\| \|AX - XA\| = \|F\|_{C_A^{(n)}(\mathbb{D})} \|\Delta_A (X)\|_{\mathcal{B}(C_A^{(n)}(\mathbb{D}))}; \end{aligned}$$

here $r(\cdot)$ denotes the spectral radius of the operator. Hence

$$\frac{1}{\|F\|_{C_A^{(n)}(\mathbb{D})}} \leq \|\Delta_A (X)\|.$$

By taking supremum over the operators X with $\|X\| \leq 1$, we have from this inequality that

$$\sup_{\|X\| \leq 1} \frac{1}{\|F_{A,X}\|_{C_A^{(n)}(\mathbb{D})}} \leq \sup_{\|X\| \leq 1} \|\Delta_A (X)\| = \|\Delta_A\|,$$

that is

$$\frac{1}{\inf_{\|X\| \leq 1} \|F_{A,X}\|_{C_A^{(n)}(\mathbb{D})}} \leq \|\Delta_A\|.$$

By denoting $C_A := \frac{1}{\inf_{\|X\| \leq 1} \|F_{A,X}\|_{C_A^{(n)}(\mathbb{D})}} > 0$, we have the desired result. The theorem is proved. \square

ACKNOWLEDGEMENTS

This work was supported by Süleyman Demirel University with Project 1918-YL-09. We are grateful to the referees for their important remarks and suggestions.

REFERENCES

1. Biswas, A., Lambert, A., and Petrovic, S. Extended eigenvalues and the Volterra operator. *Glasgow Math. J.*, 2002, **44**, 521–534.
2. Bourdon, P. S. and Shapiro, J. H. Intertwining relations and extended eigenvalues for analytic Toeplitz operators. *Ill. J. Math.*, 2008, **52**, 1007–1030.
3. Domanov, I. Yu. and Malamud, M. M. On the spectral analysis of direct sums of Riemann–Liouville operators in Sobolev spaces of vector functions. *Int. Equat. Oper. Theory*, 2009, **63**, 181–215.
4. Gürdal, M. Description of extended eigenvalues and extended eigenvectors of integration operator on the Wiener algebra. *Expo. Math.*, 2009, **27**, 153–160.
5. Gürdal, M. On the extended eigenvalues and extended eigenvectors of shift operator on the Wiener algebra. *Appl. Math. Lett.*, 2009, **22**, 1727–1729.
6. Karaev, M. T. Invariant subspaces, cyclic vectors, commutant and extended eigenvectors of some convolution operators. *Methods Functional Anal. Topology*, 2005, **11**, 48–59.
7. Karaev, M. T. On some applications of ordinary and extended Duhamel products. *Siberian Math. J.*, 2005, **46**, 431–442 (translated from *Sibirsk. Mat. Zh.*, 2005, **46**, 553–566).
8. Karaev, M. T. and Saltan, S. A Banach algebra structure for the Wiener algebra $W(\mathbb{D})$ of the disc. *Complex Variables Theory Appl.*, 2005, **50**, 299–305.
9. Malamud, M. M. Similarity of Volterra operators and related problems in the theory of differential equations of fractional orders. *Trans. Moscow Math. Soc.*, 1995, **56**, 57–122 (translated from *Trudy Moskov. Mat. Obshch.*, 1994, **55**, 73–148).

10. Malamud, M. M. Invariant and hyperinvariant subspaces of direct sums of simple Volterra operators. *Oper. Theory Adv. Appl.*, 1998, **102**, 143–167.
11. Nagnibida, N. I. Description of commutants of integration operator in analytic spaces. *Siberian Math. J.*, 1981, **22**, 748–752 (translated from *Sibirsk. Mat. Zh.*, 1981, **22**, 127–131).
12. Tkachenko, V. A. Invariant subspaces and unicellularity of operators of generalized integration in spaces of analytic functionals. *Math. Notes*, 1997, **22**, 221–230.
13. Wigley, N. M. The Duhamel product of analytic functions. *Duke Math. J.*, 1974, **41**, 211–217.

Mõnedest operaatorvõrranditest analüütiliste funktsioonide ruumis

Mehmet Gürdal ja Filiz Şöhret

On uuritud teatavate konvolutsioonioperaatorite laiendatud omaväärtusi, laiendatud omavektoreid ja tsükli-vektoreid ning nendega seonduvaid küsimusi.