Tallinna Tehnikaülikool
Mehhatroonikainstituut

Jüri Kirs, Kalju Kenk

Kodutöö S-14

Tasapinnalise kujundi raskuskeskme leidmine

Tallinn 2004

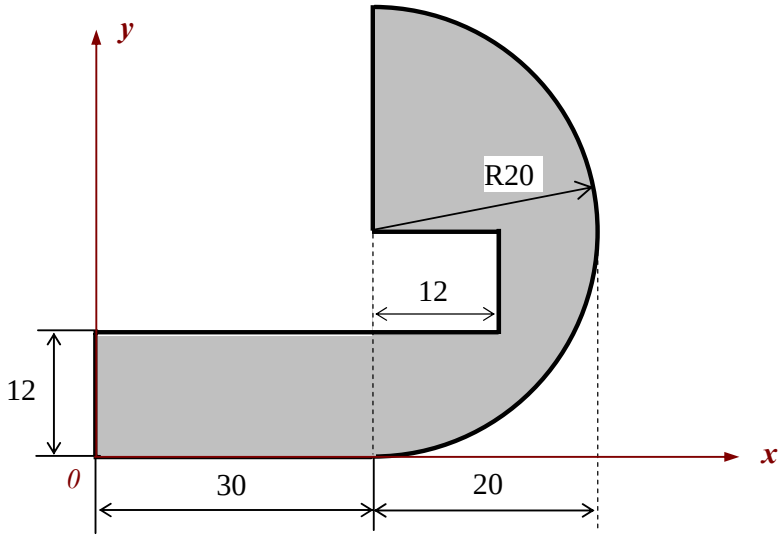
ISBN 9985-894-79-0

Kodutöö S-14

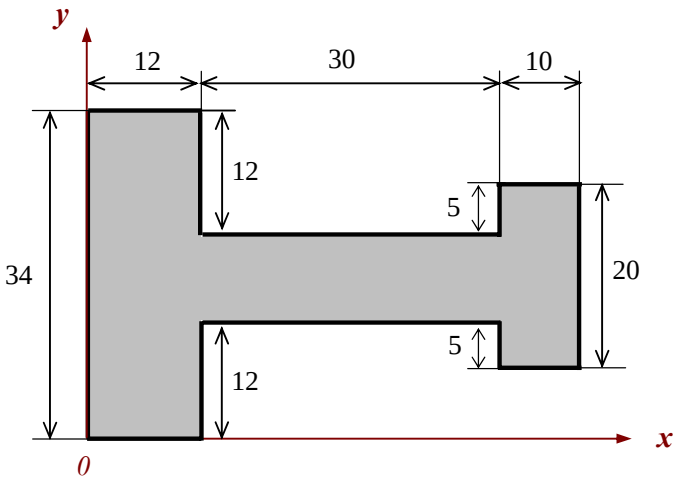
Tasapinnalise kujundi raskuskeskme leidmine

Leida alljärgnevate kujundite raskuskeskmed. Kõik vajalikud arvulised andmed on toodud vastava variandi juures. Mõõdud on antud sentimeetrites. Kujundite järel on toodud ka kaks näiteülesannet. Raskuskeskmete leidmisest võib veel lugeda Endel Topnik'u õpikus „Insenerimehaanika ülesannetest I. Staatika", Tallinn 2001, leheküljelt 105 ning ka S. Targ'i õpikus "Teoreetiline mehaanika", Tallinn 1966, lehekülgedelt 124-125.

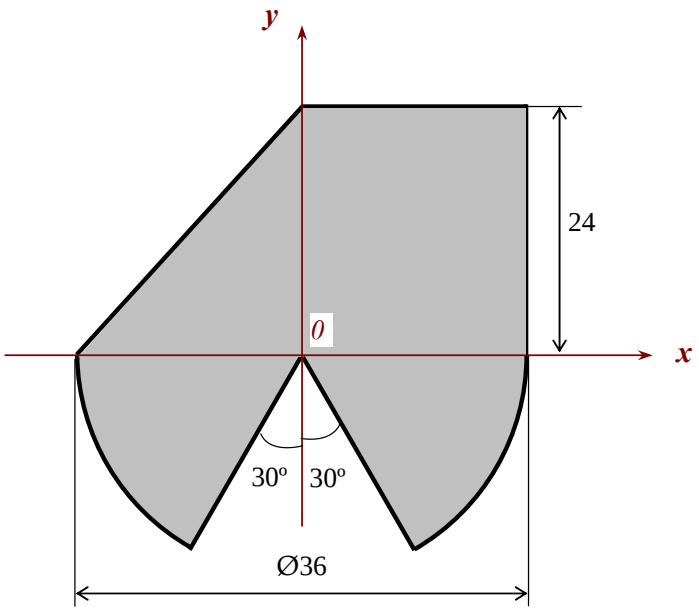
Variant 1.



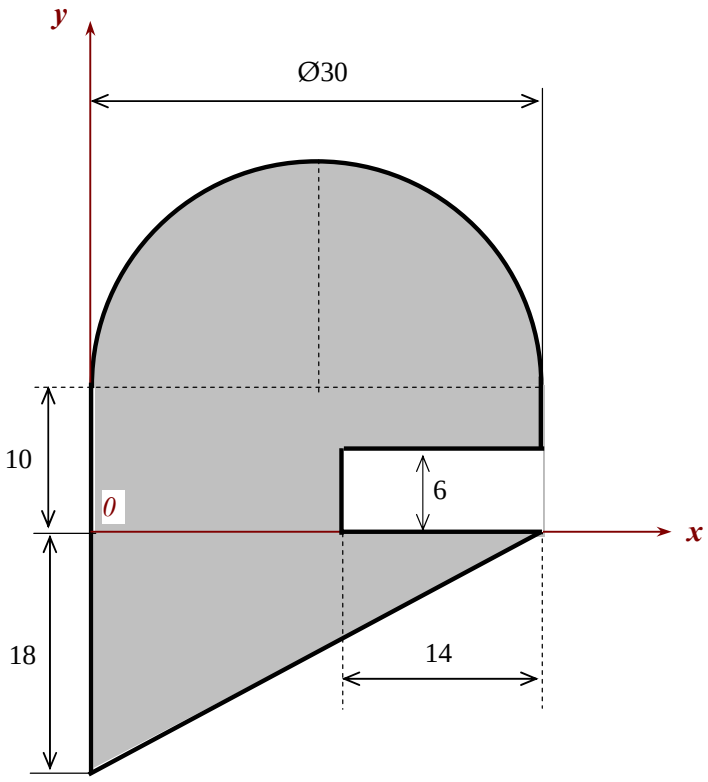
Variant 2.



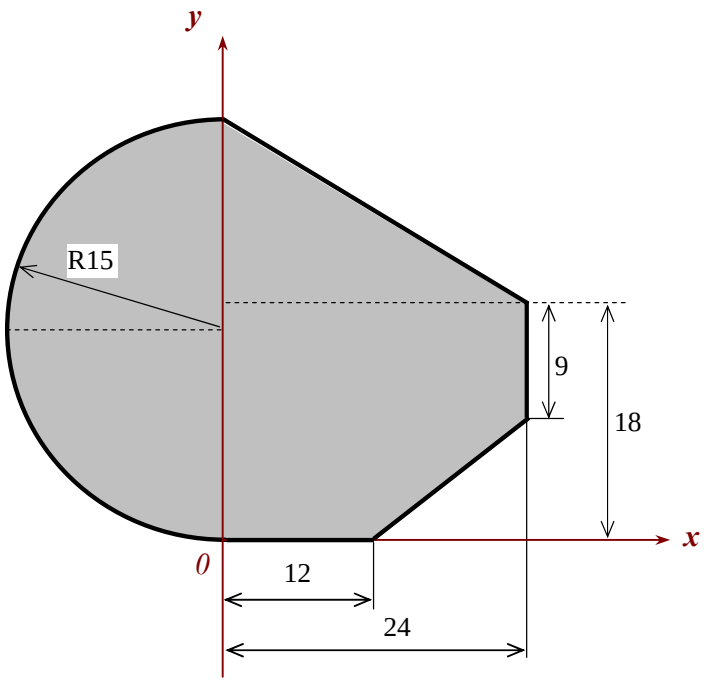
Variant 3.



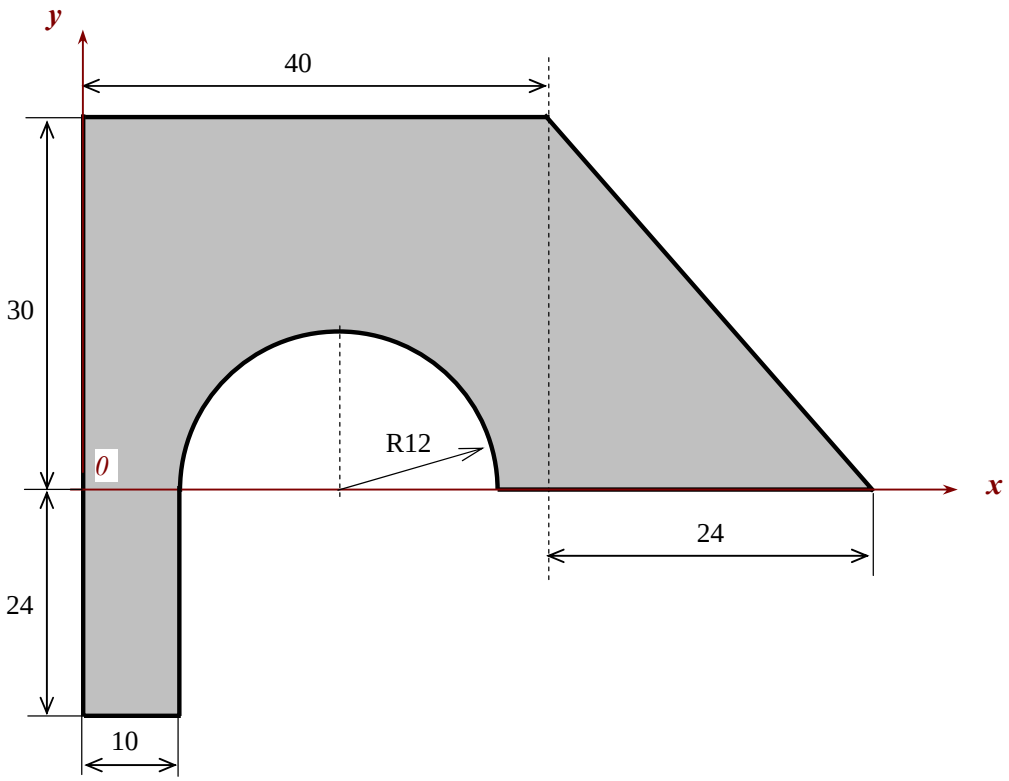
Variant 4.



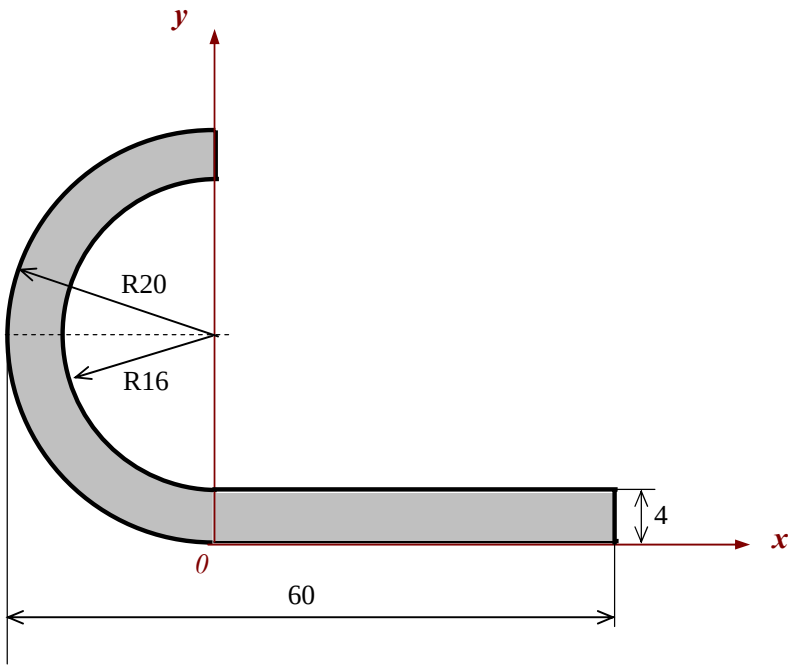
Variant 5.



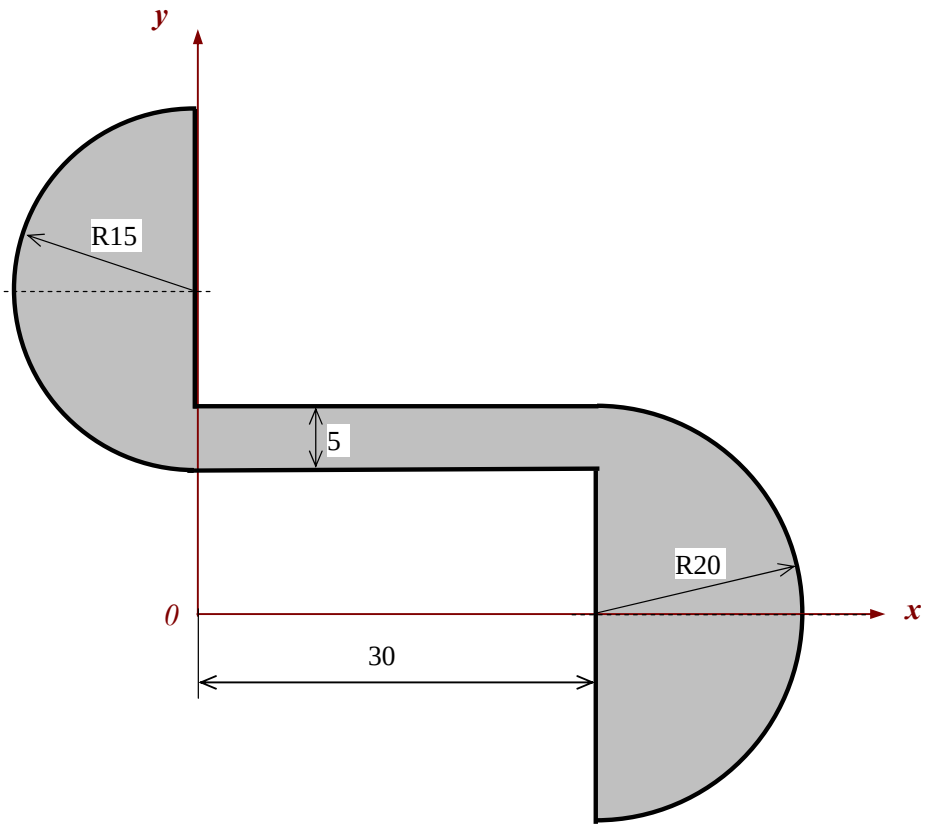
Variant 6.



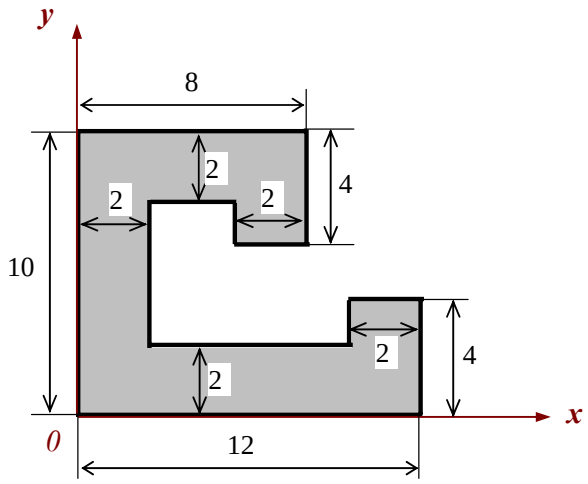
Variant 7.



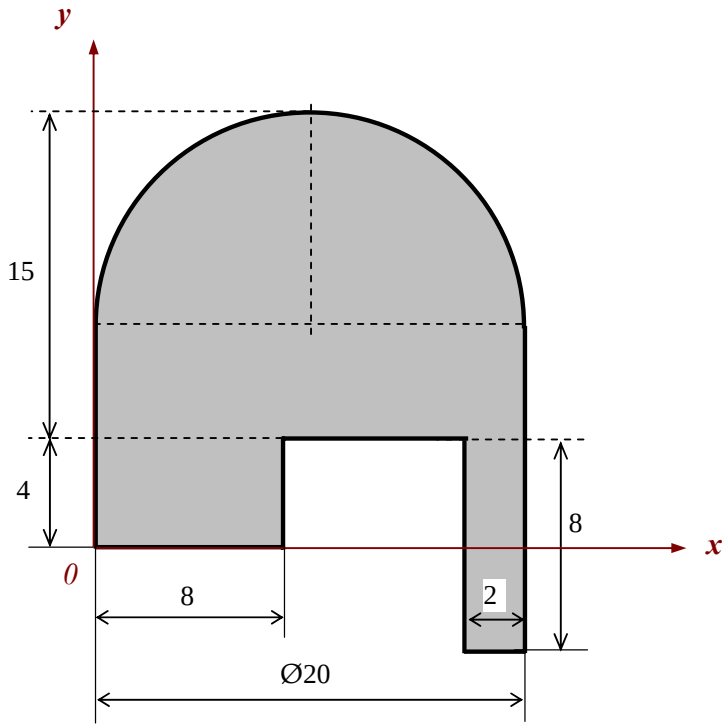
Variant 8.



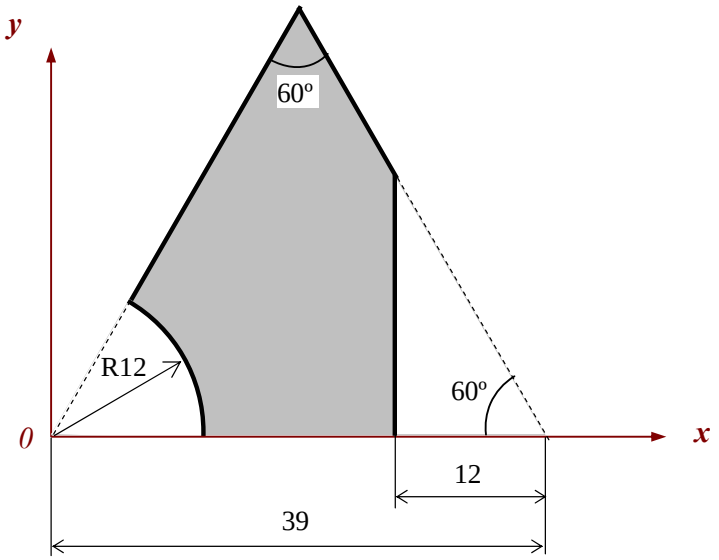
Variant 9.



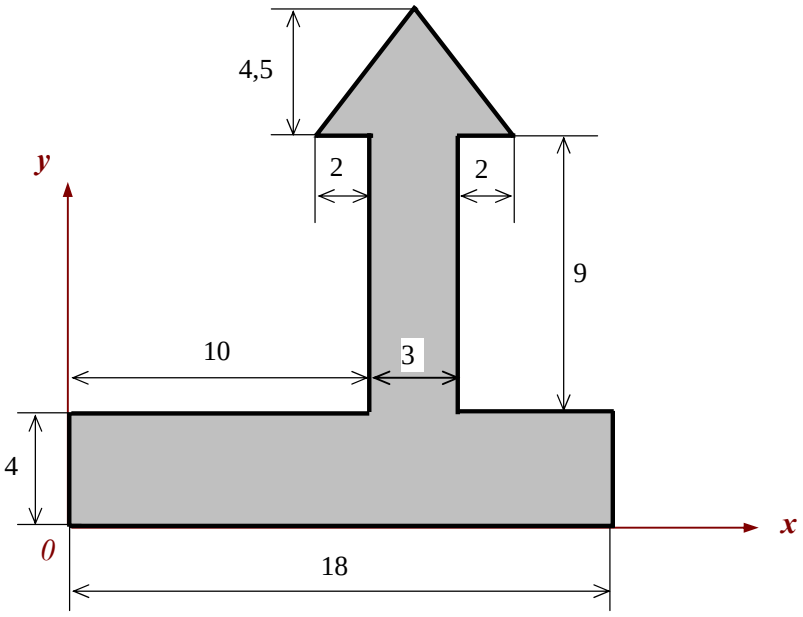
Variant 10.



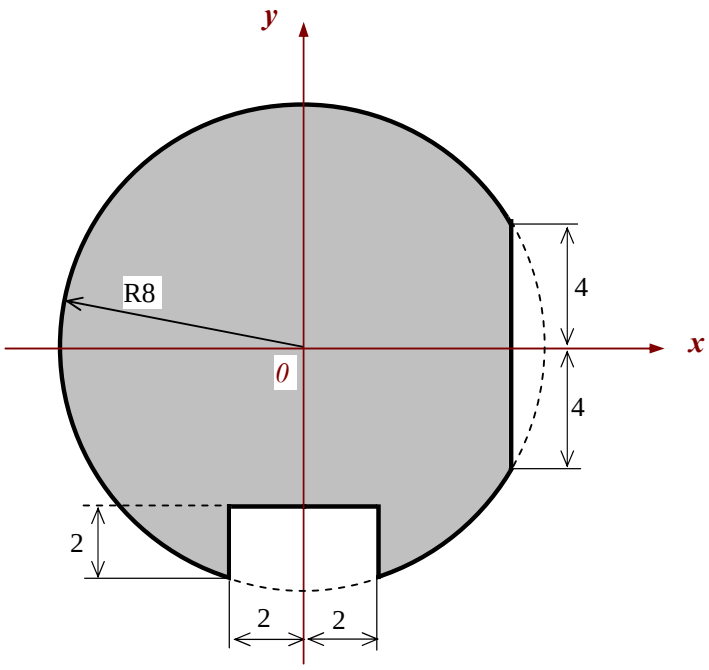
Variant 11.



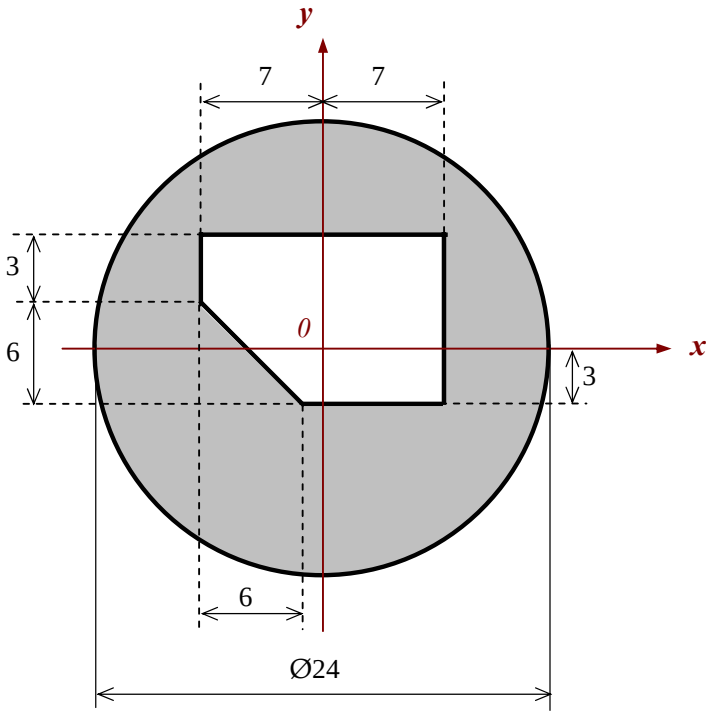
Variant 12.



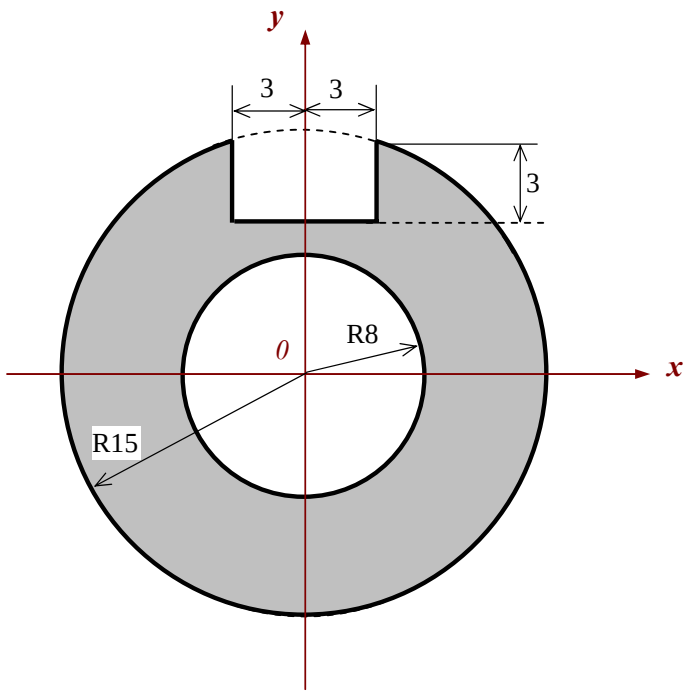
Variant 13.



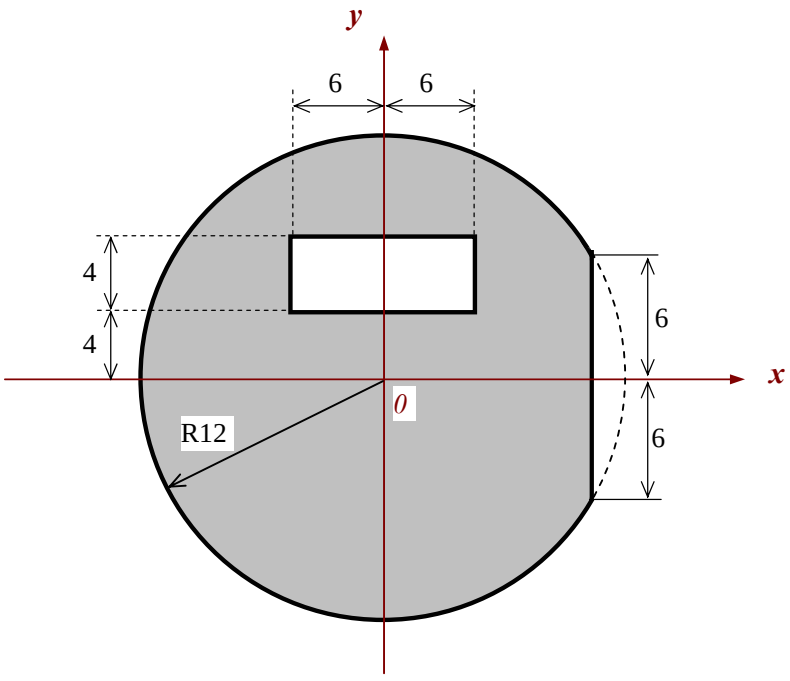
Variant 14.



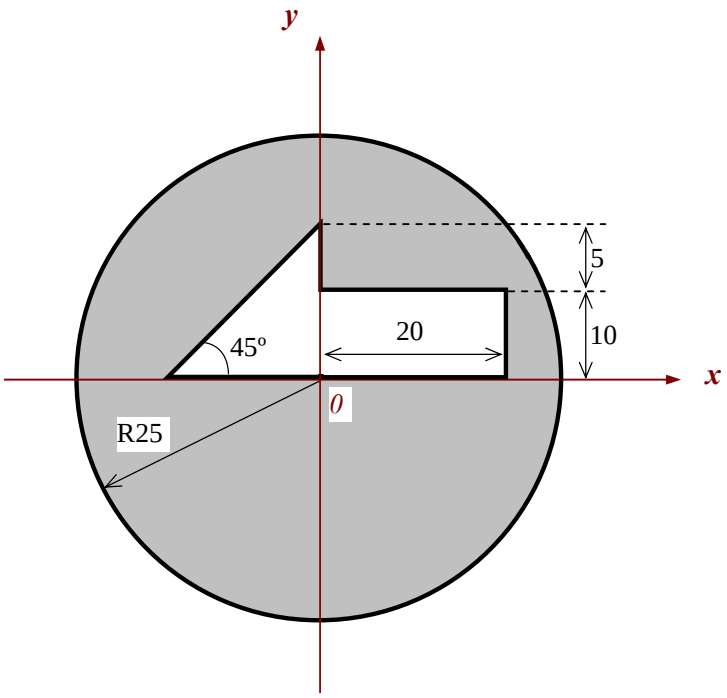
Variant 15.



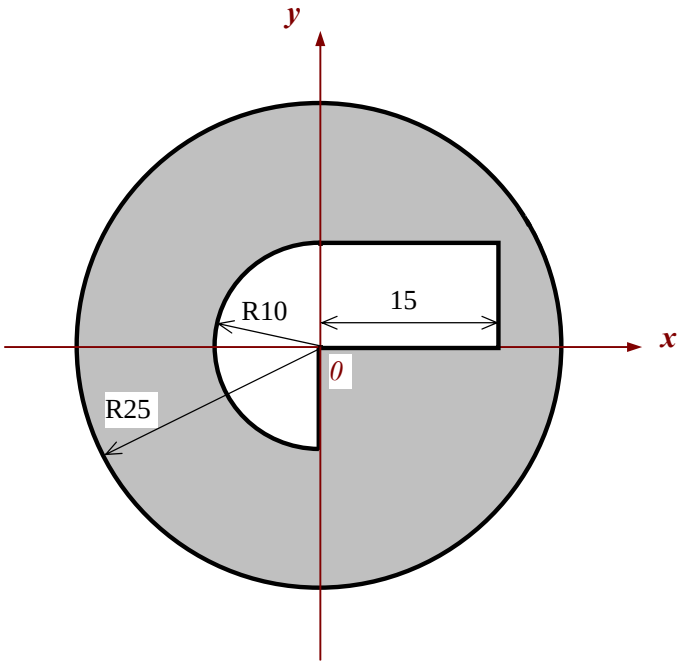
Variant 16.



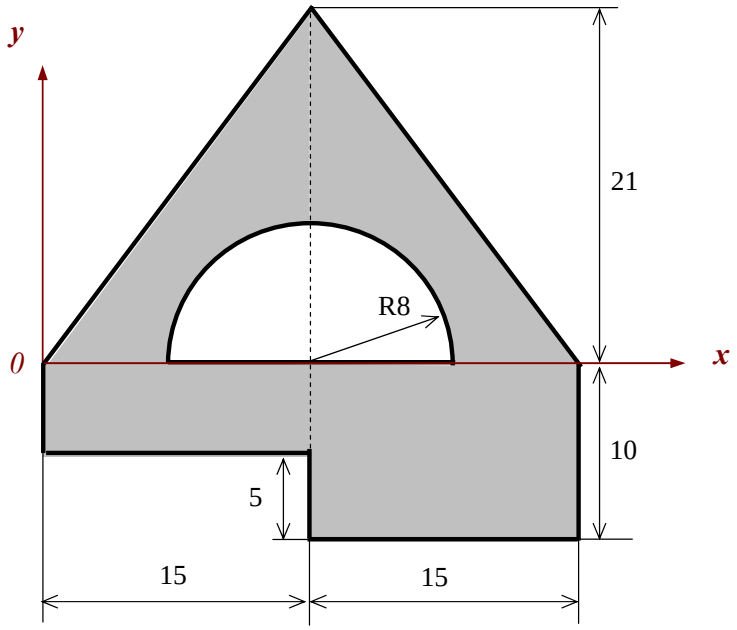
Variant 17.



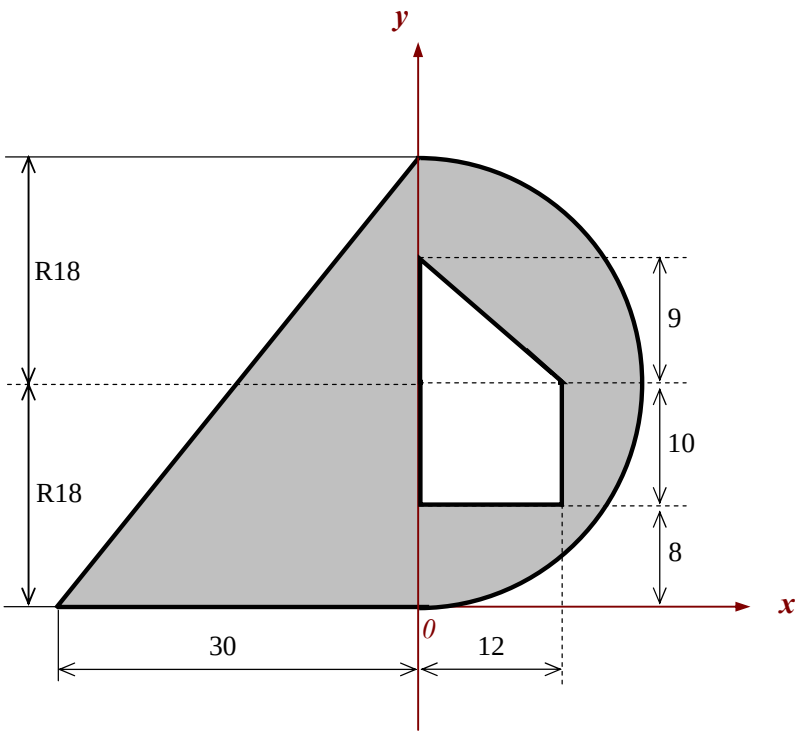
Variant 18.



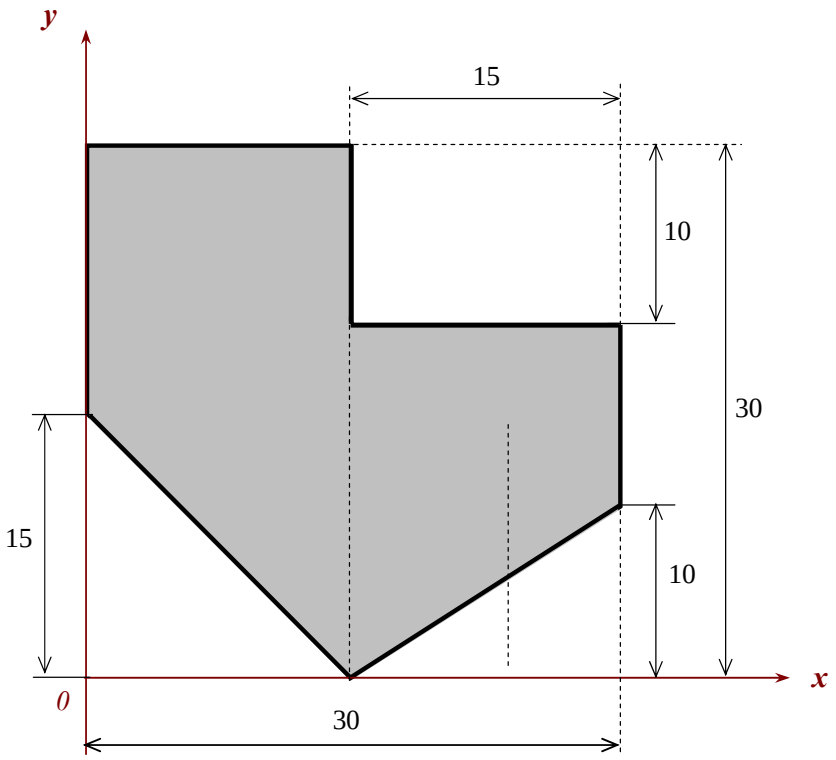
Variant 19.



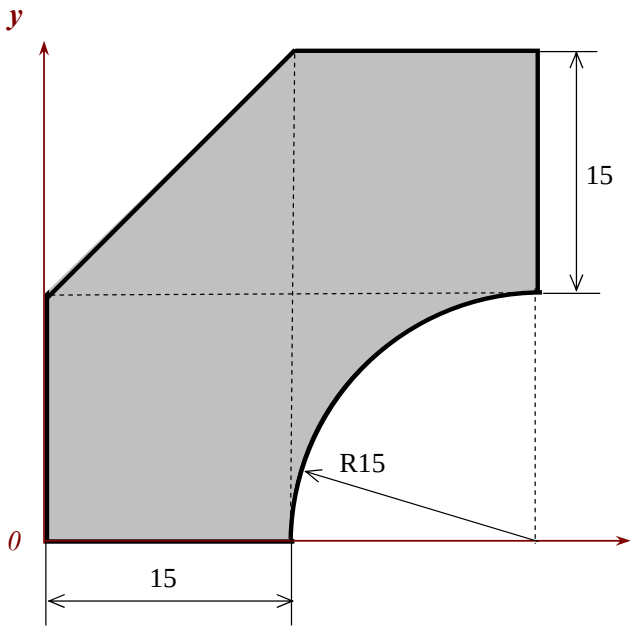
Variant 20.



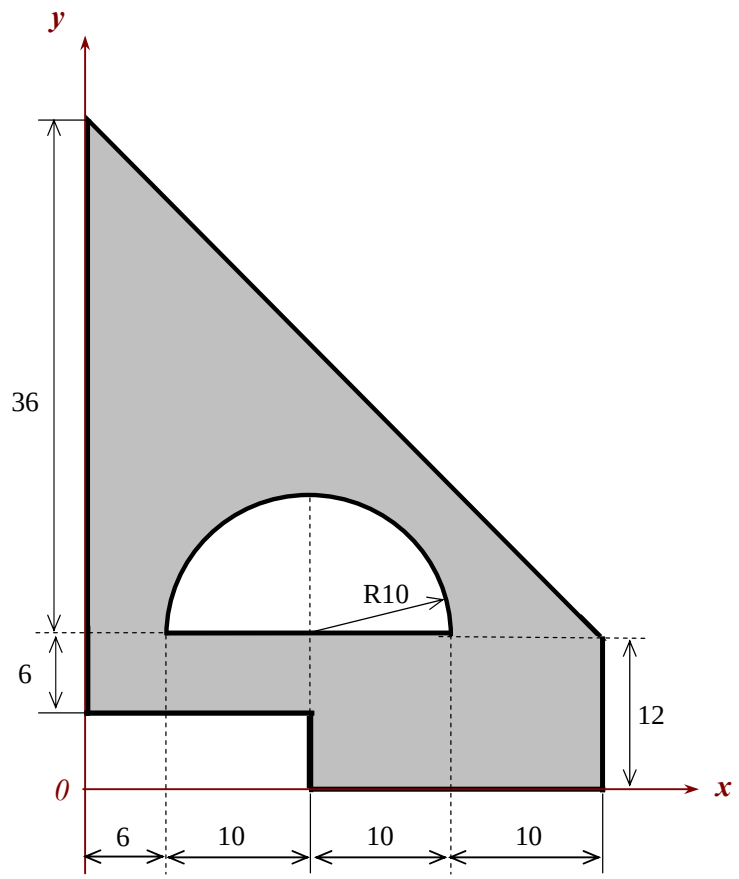
Variant 21.



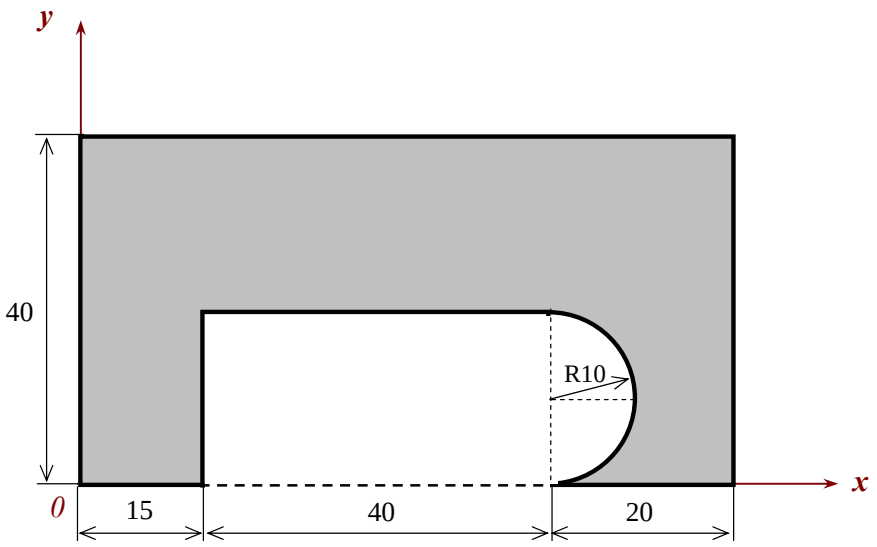
Variant 22.



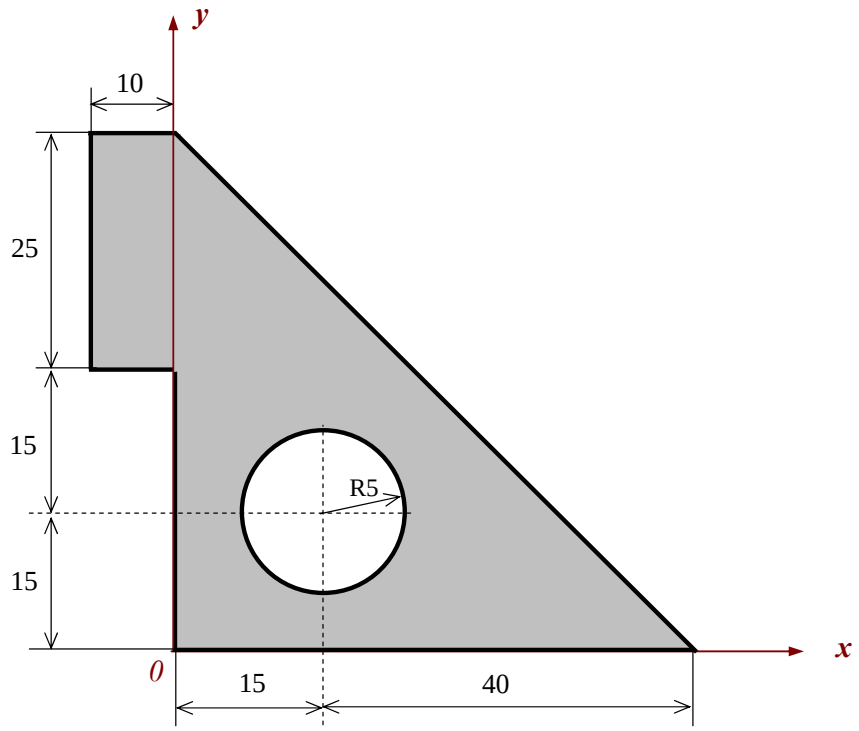
Variant 23.



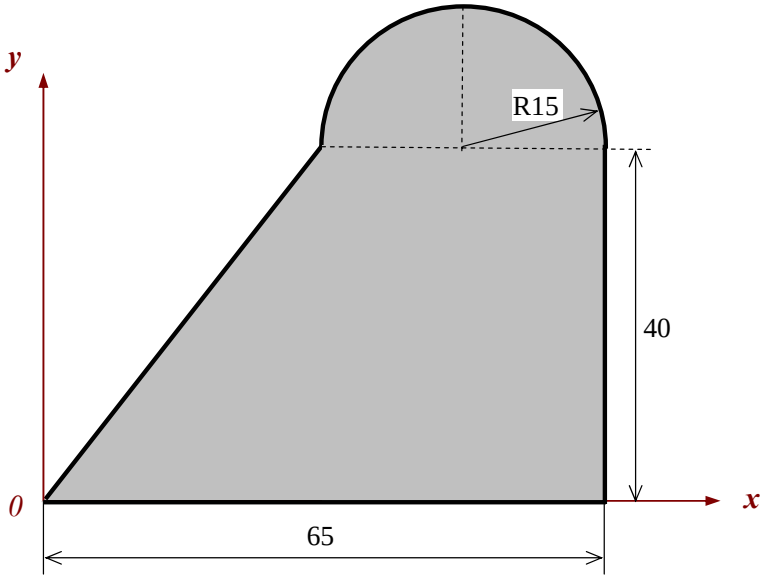
Variant 24.



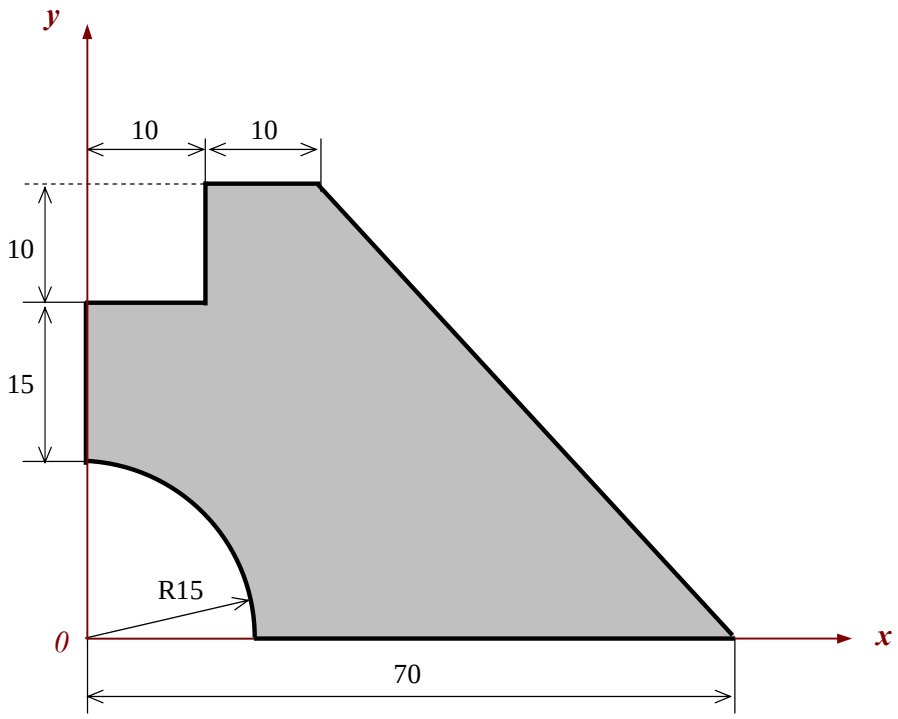
Variant 25.



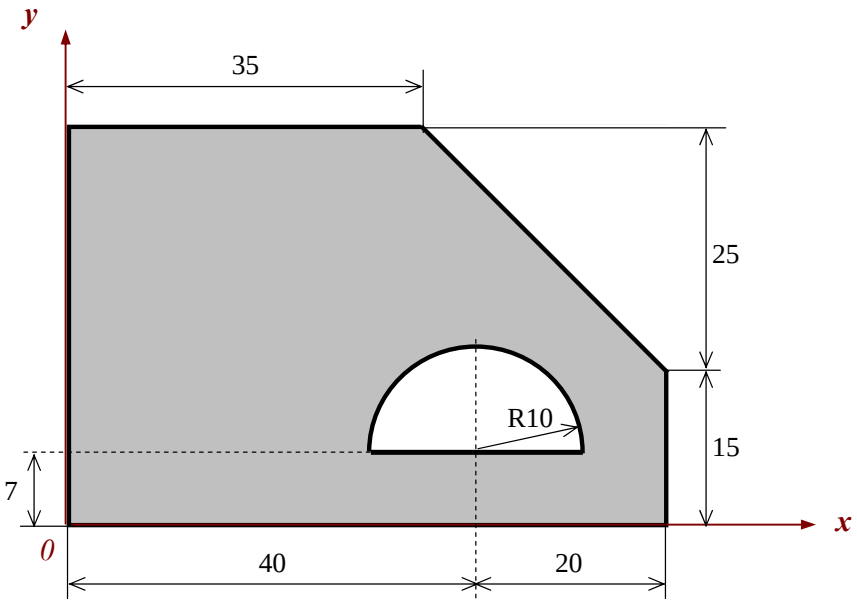
Variant 26.



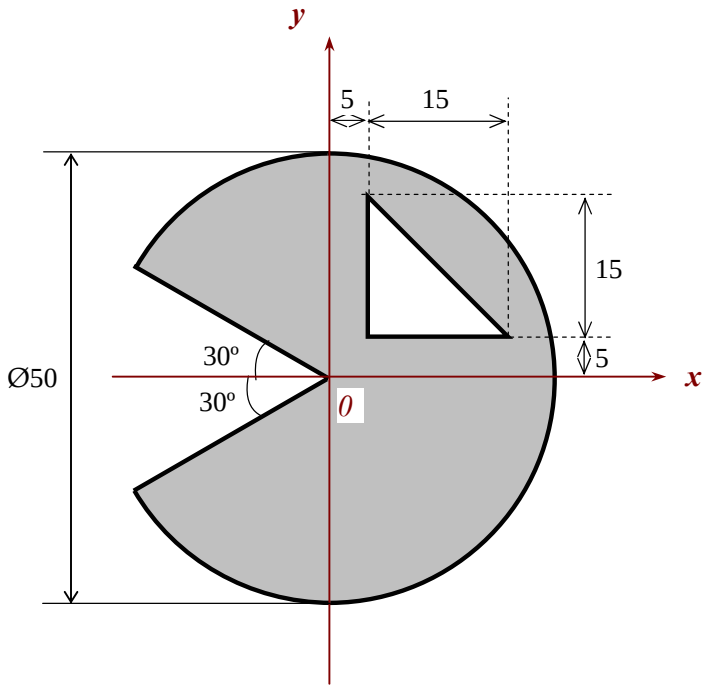
Variant 27.



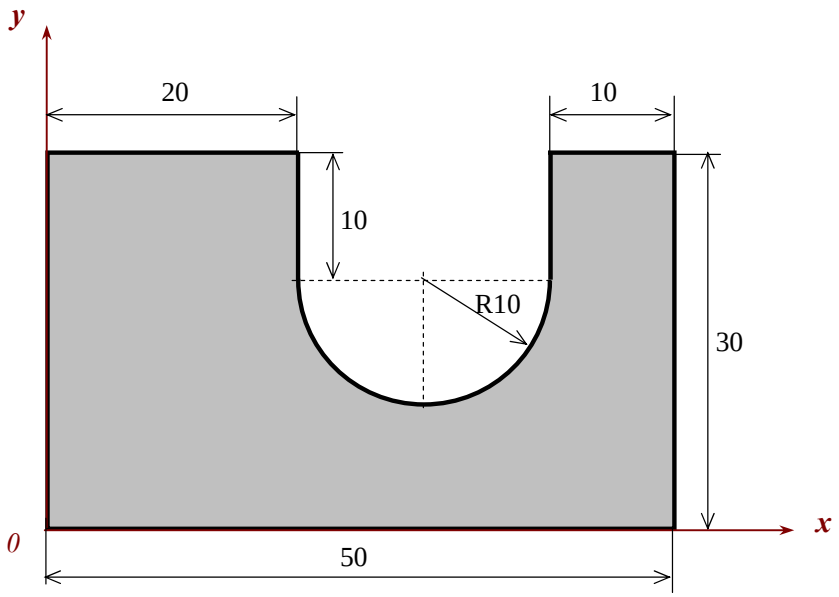
Variant 28.



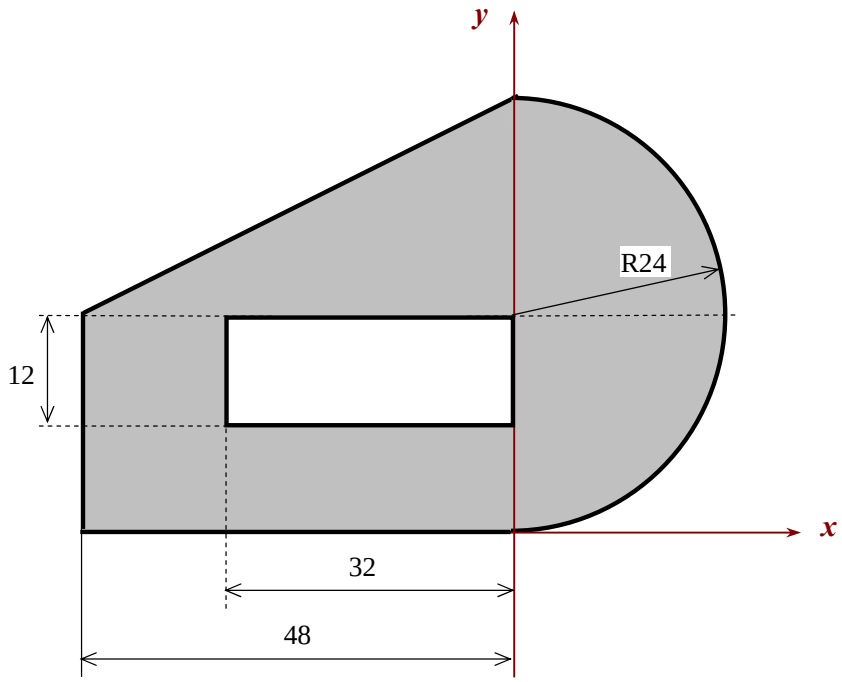
Variant 29.



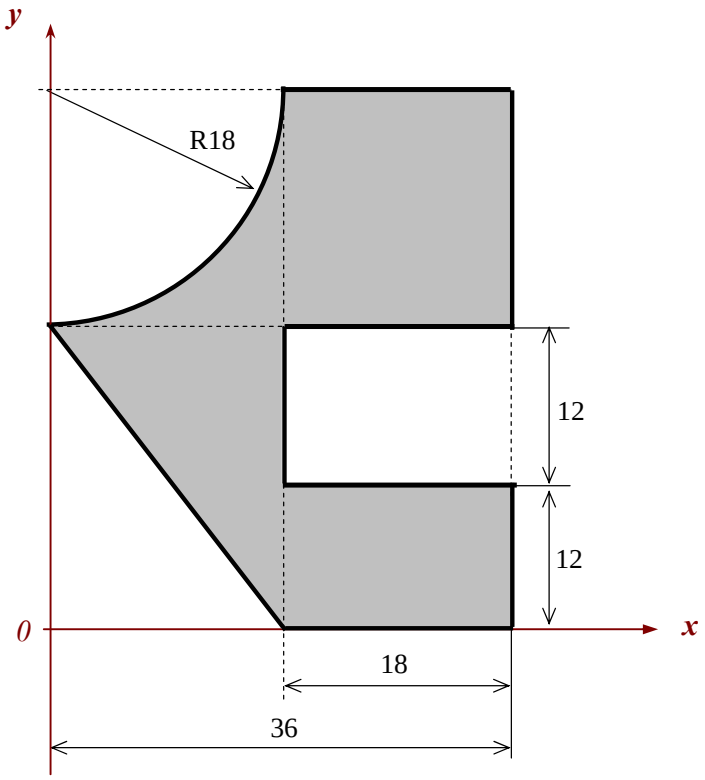
Variant 30.



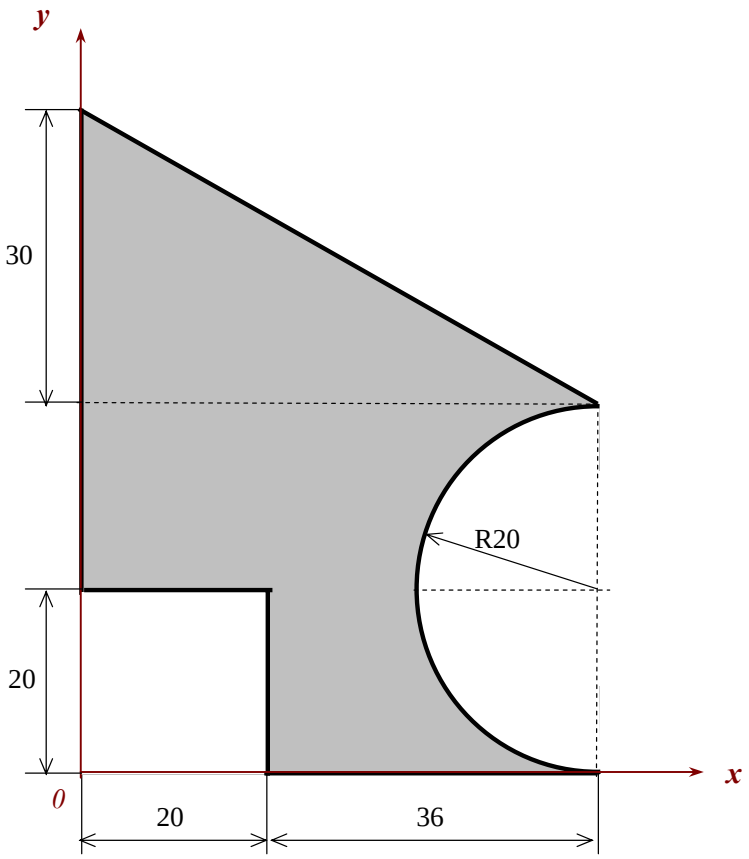
Variant 31.



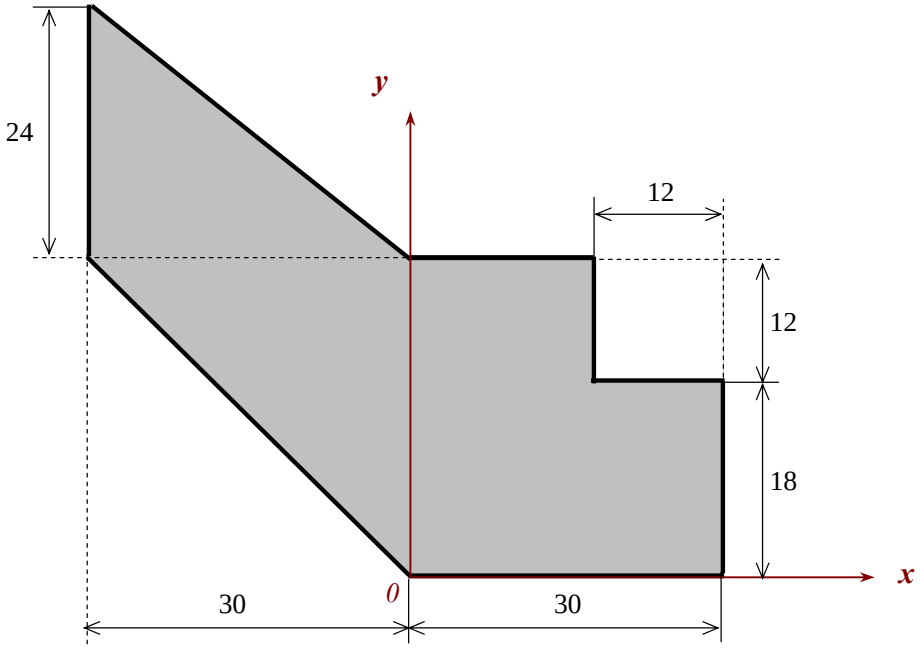
Variant 32.



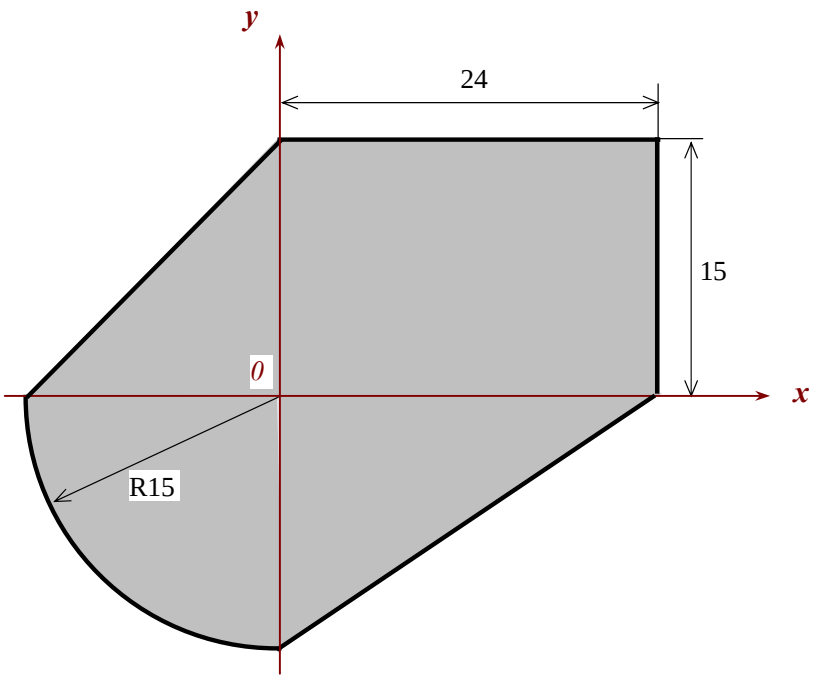
Variant 33.



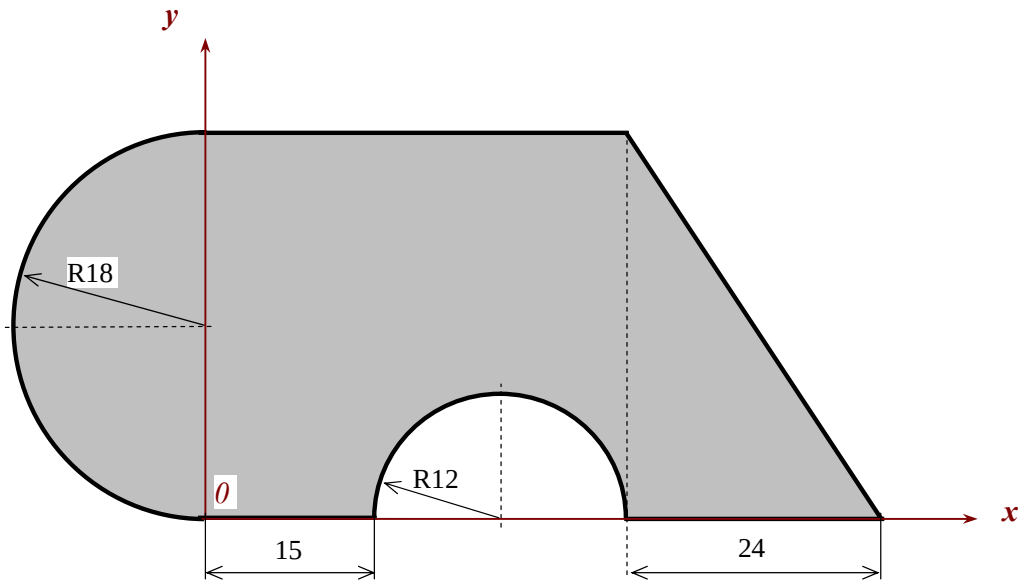
Variant 34.



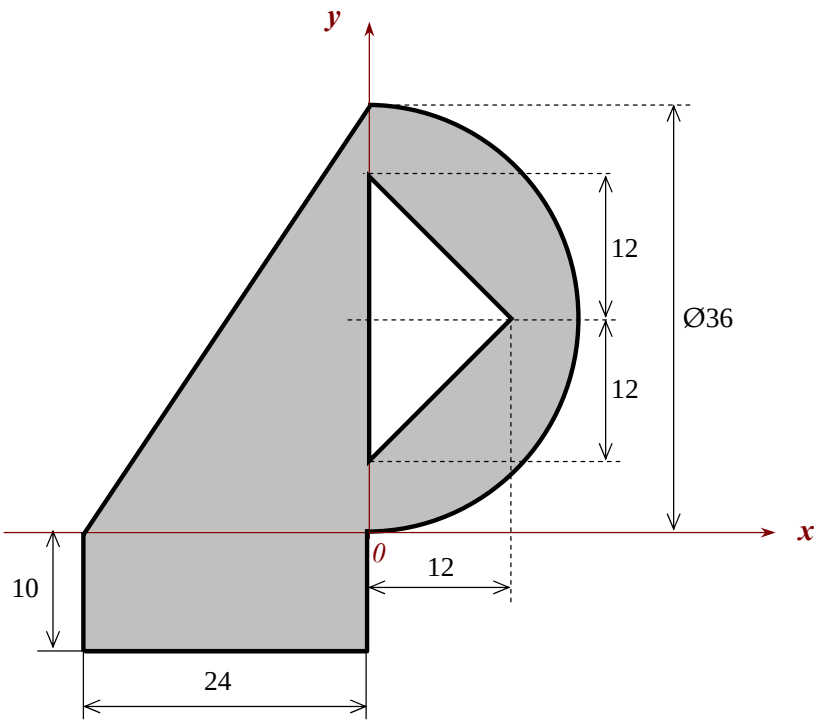
Variant 35.



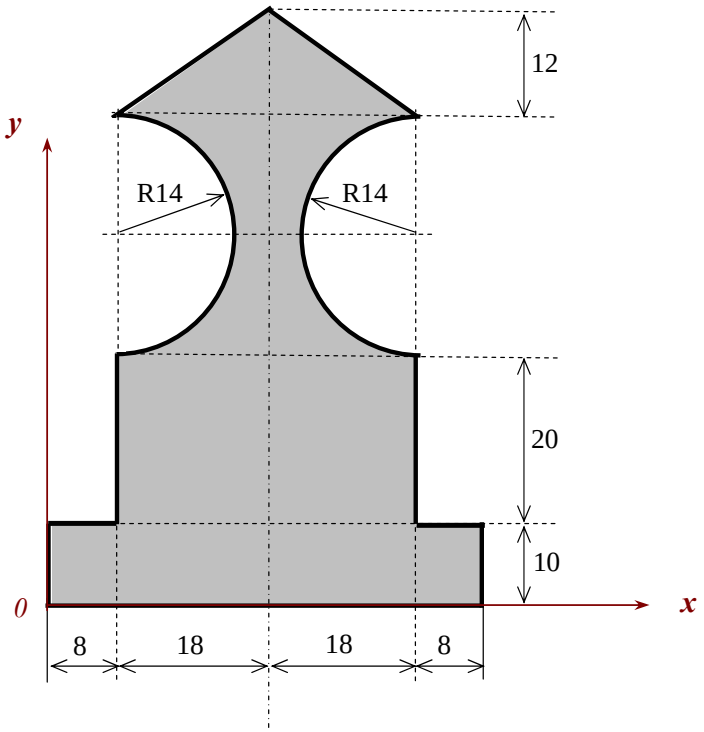
Variant 36.



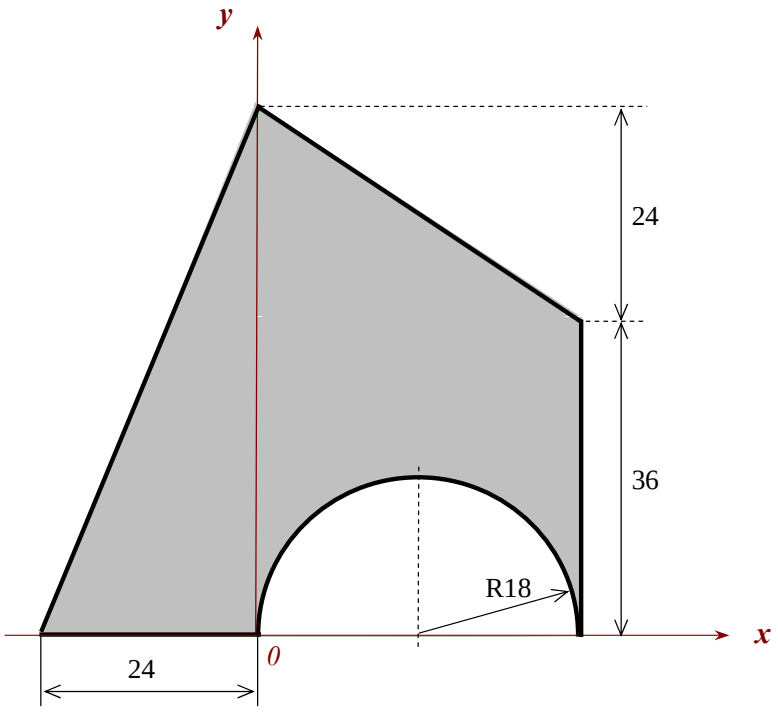
Variant 37.



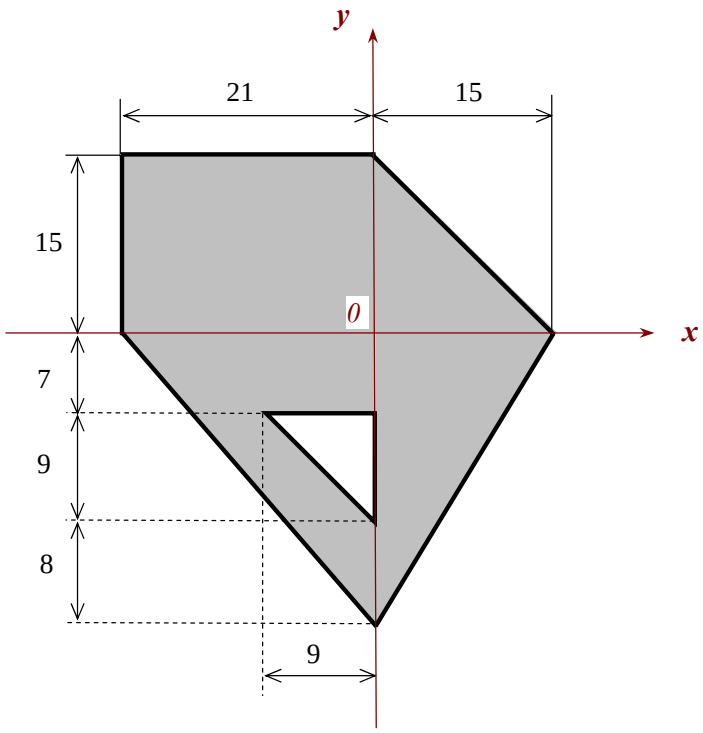
Variant 38.



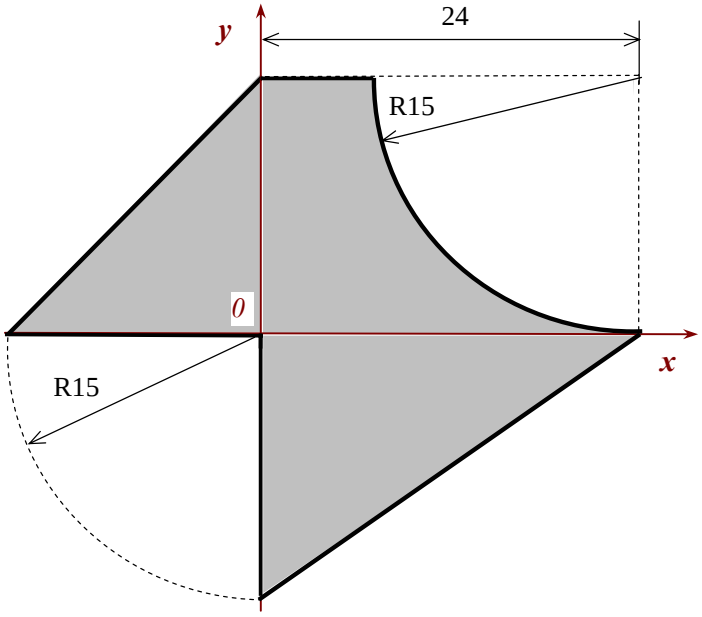
Variant 39.



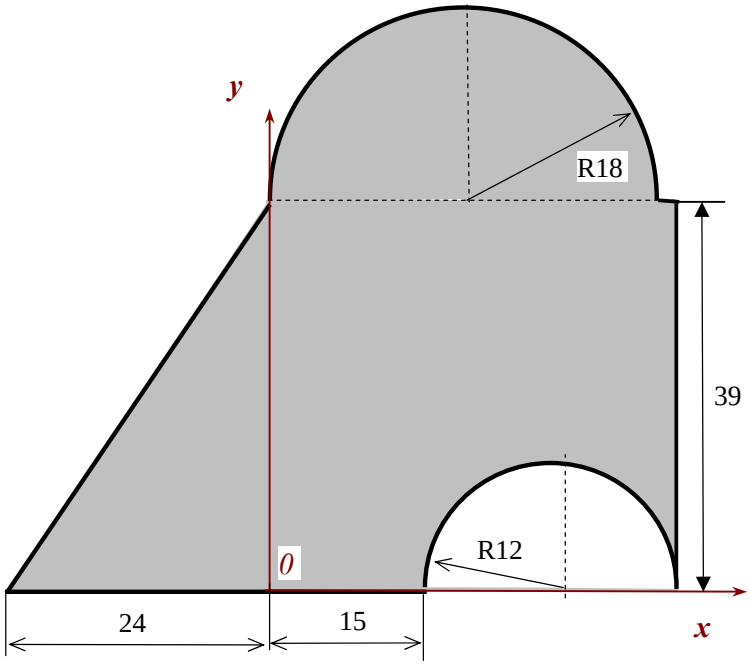
Variant 40.



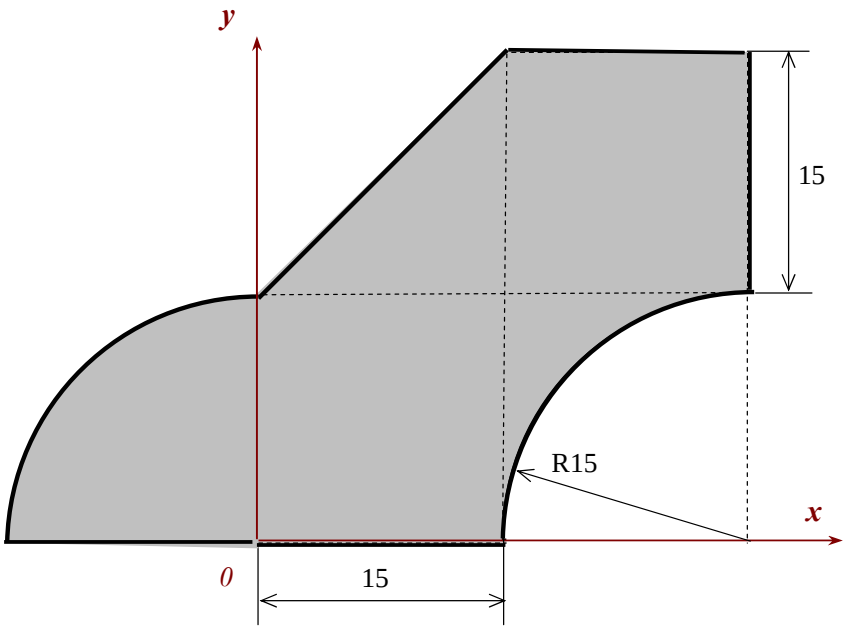
Variant 41.



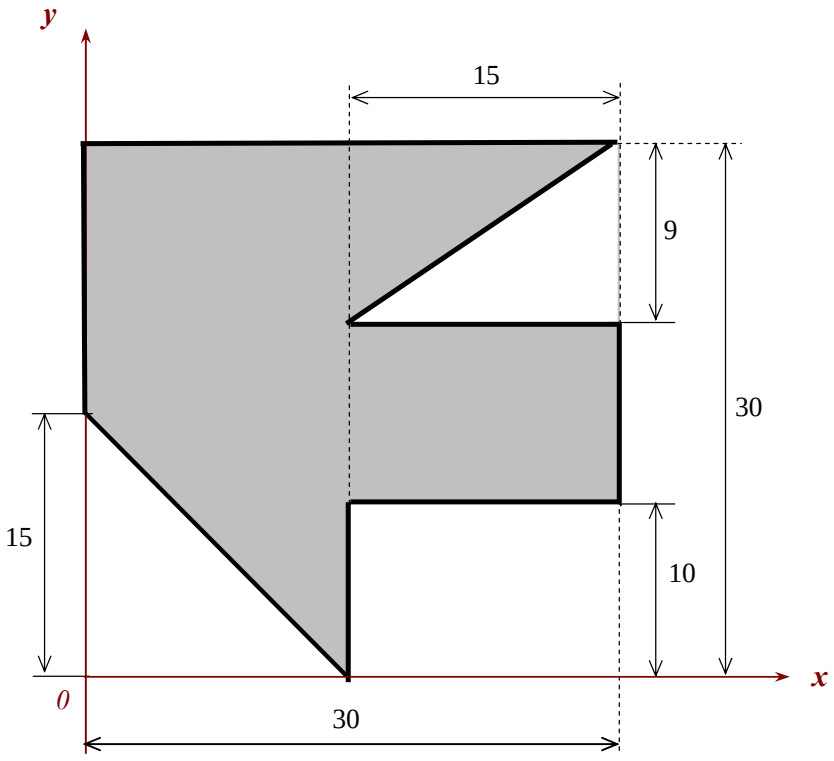
Variant 42



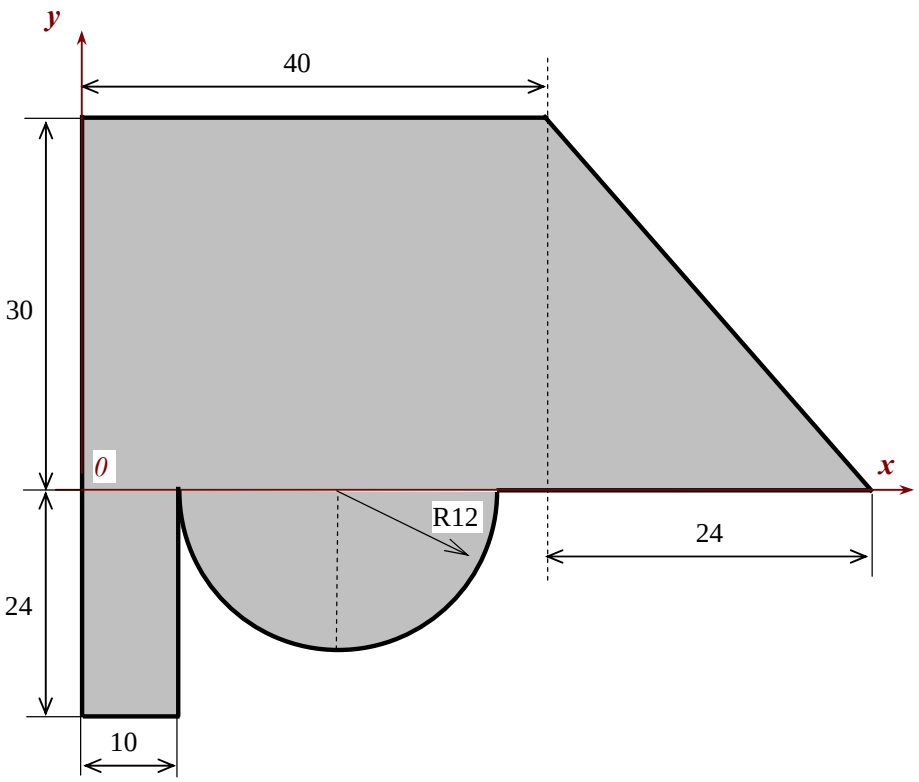
Variant 43.



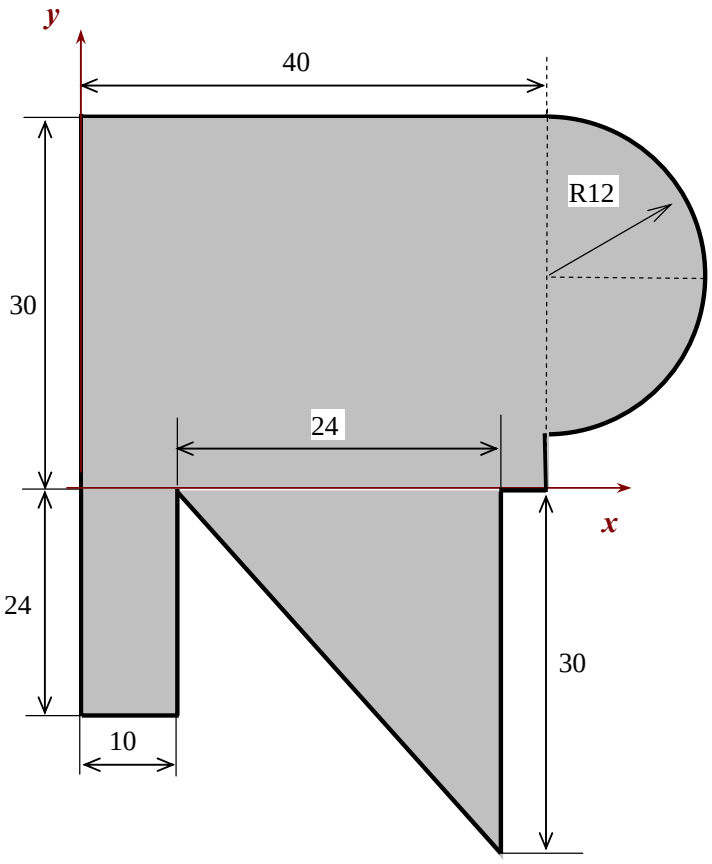
Variant 44.



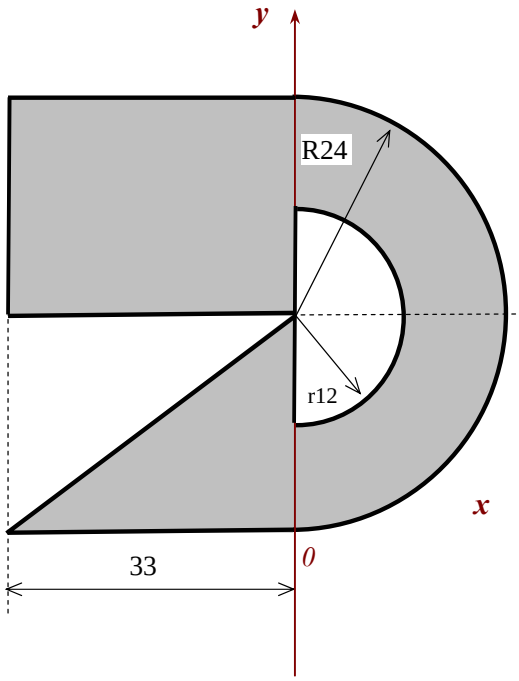
Variant 45.



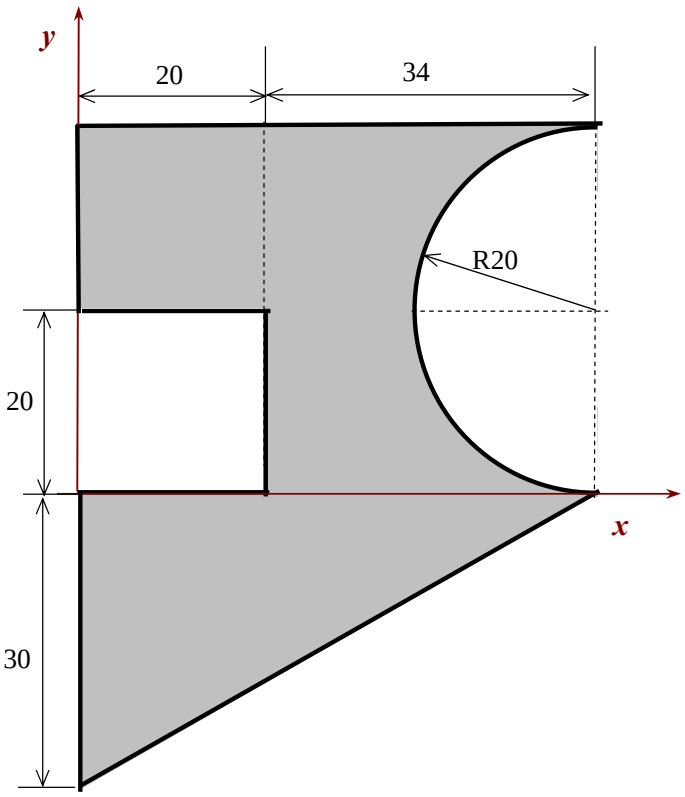
Variant 46.



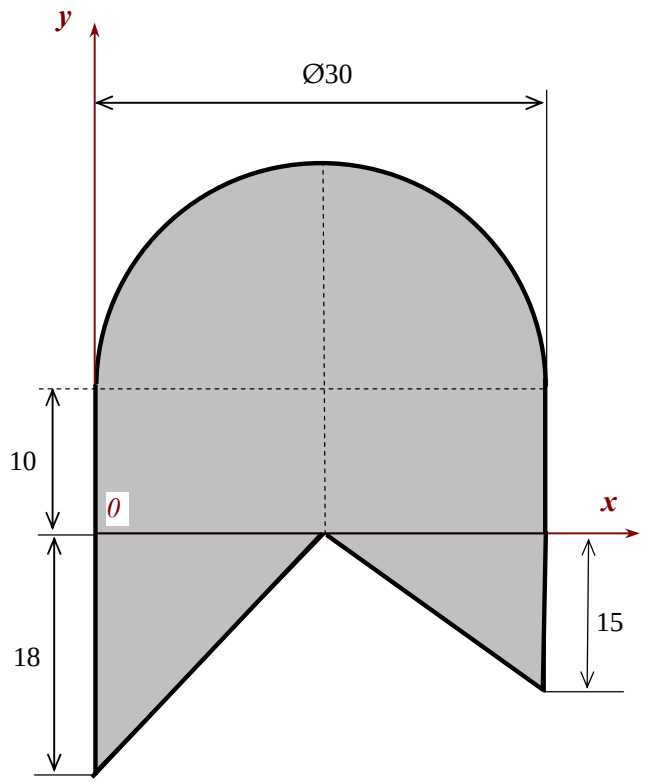
Variant 47.



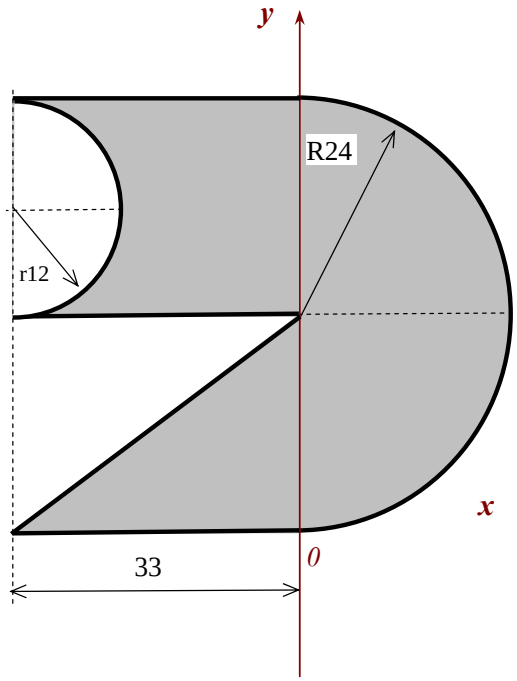
Variant 48.



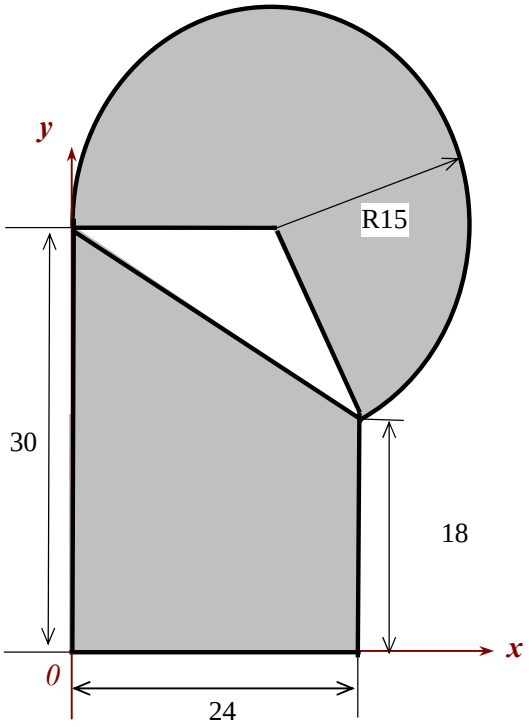
Variant 49.



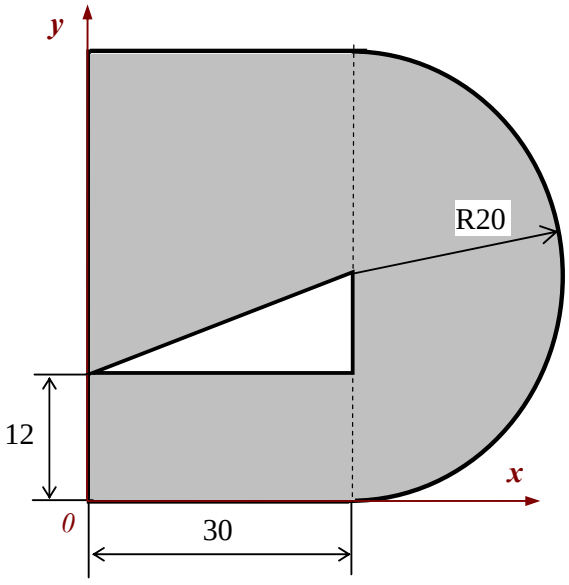
Variant 50.



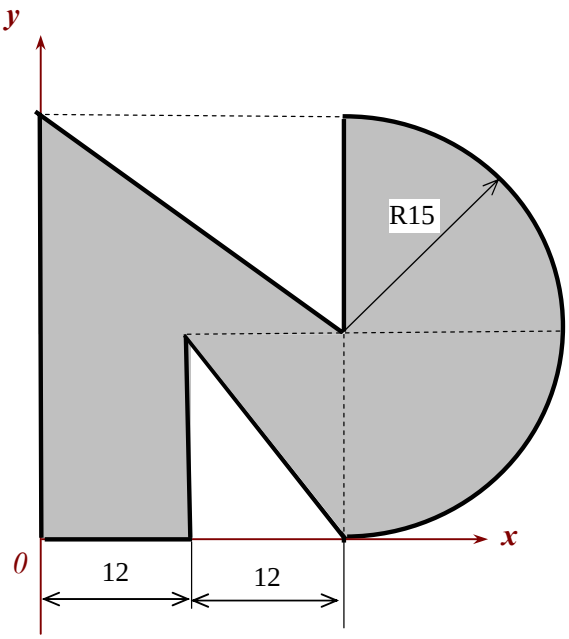
Variant 51.



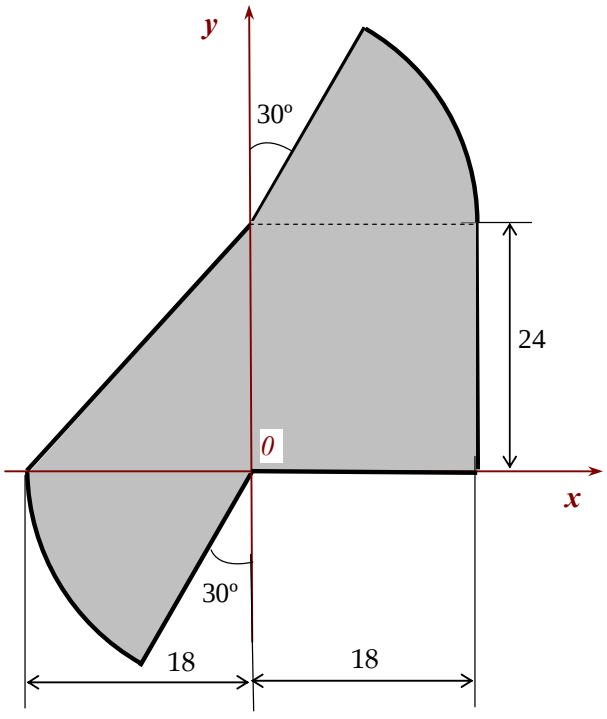
Variant 52.



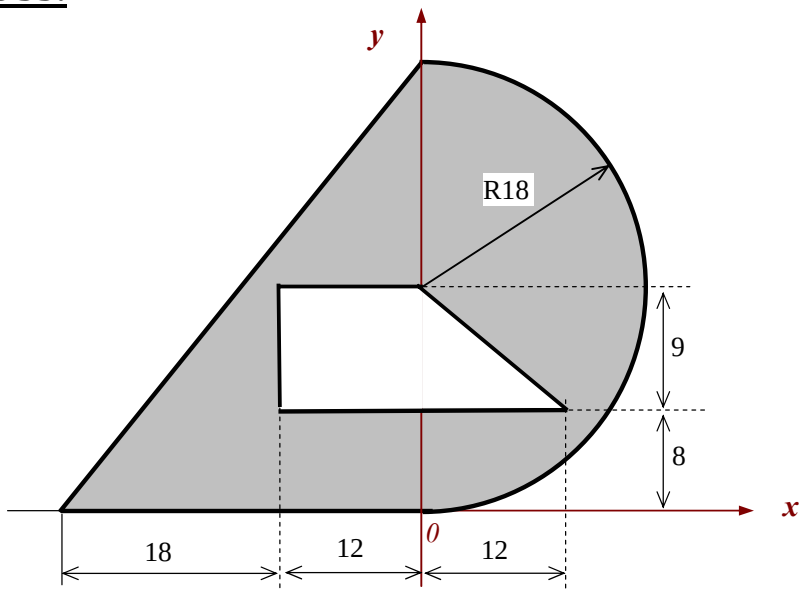
Variant 53.



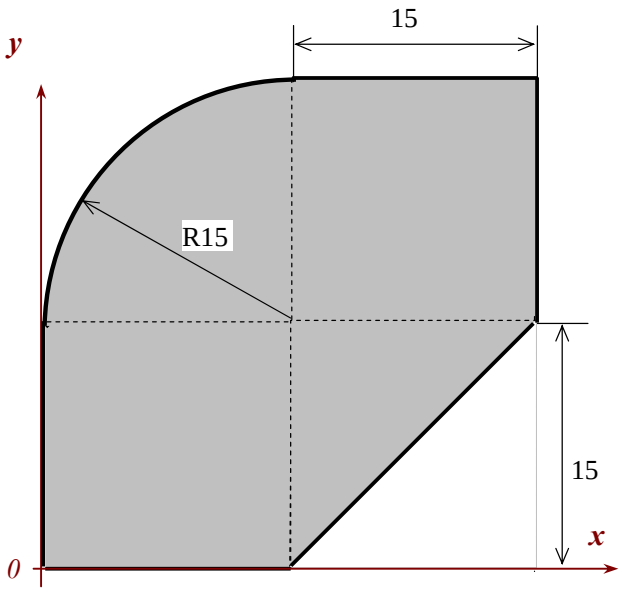
Variant 54.



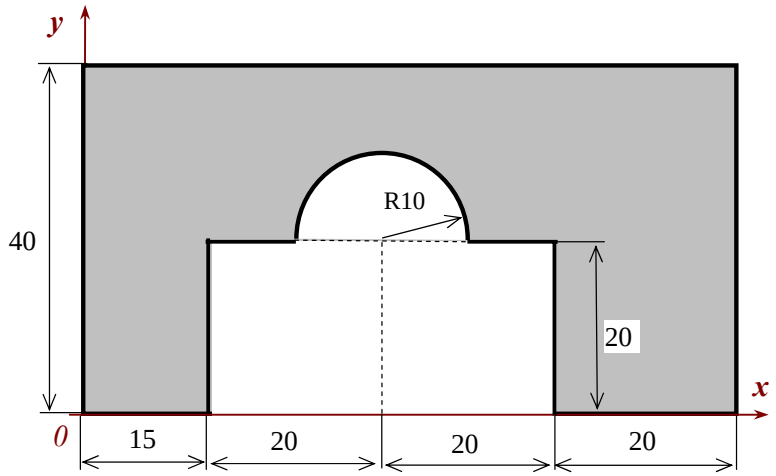
Variant 55.



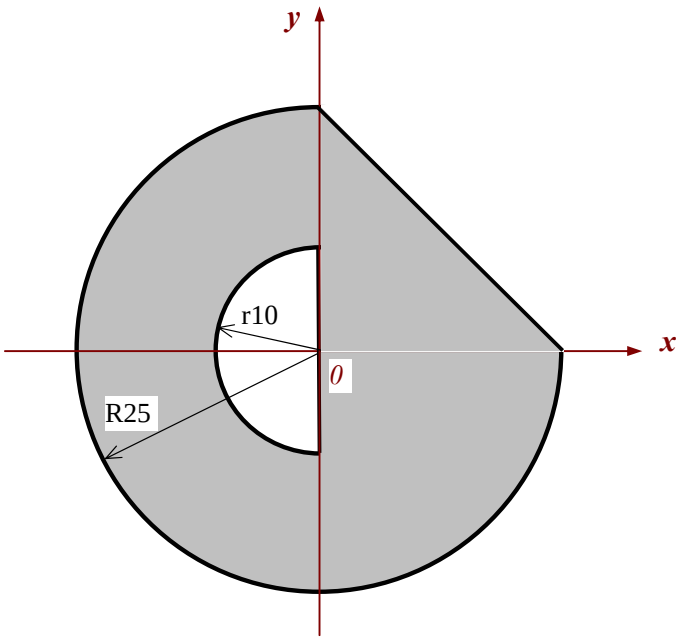
Variant 56.



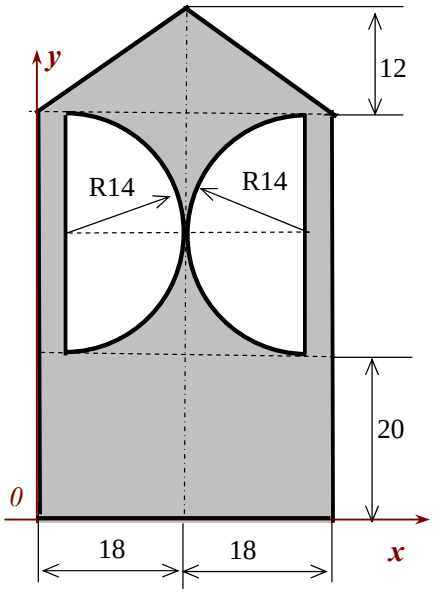
Variant 57



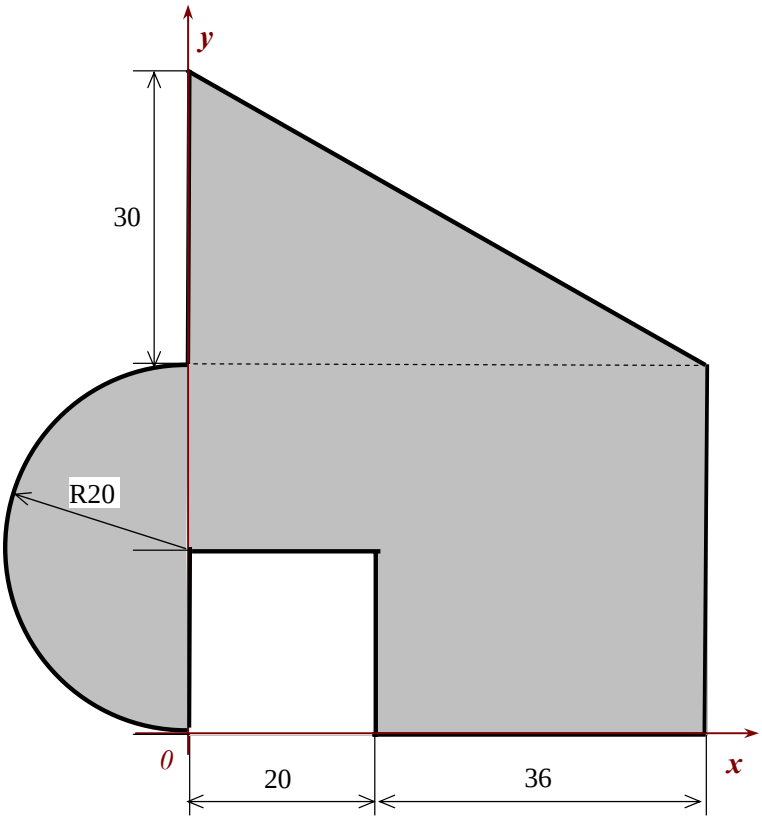
Variant 58



Variant 59

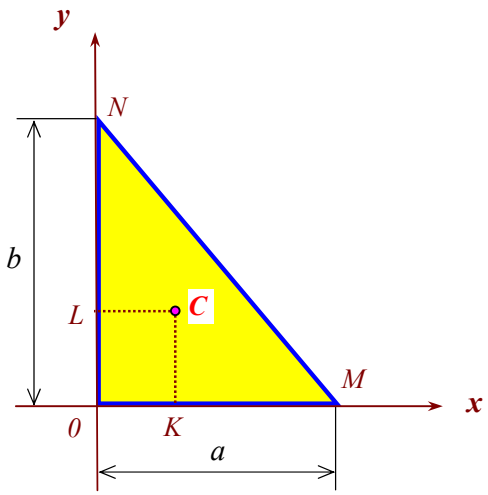


Variant 60



Mõningate kujundite raskuskeskme C koordinaadid ja pindala

1. Täisnurkne kolmnurk.



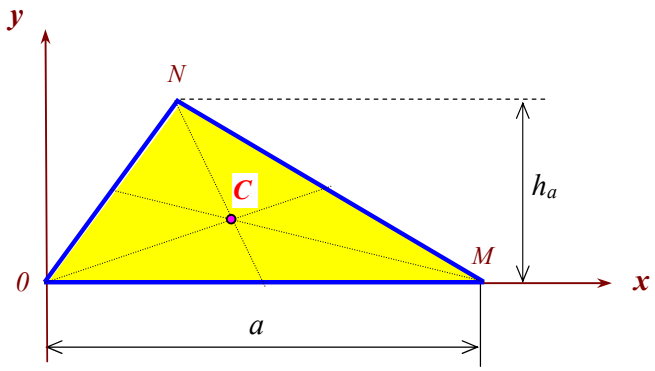
$$OK = LC = \frac{a}{3} \tag{1.1}$$

$$OL = KC = \frac{b}{3} \tag{1.2}$$

Pindala: $A = \frac{ab}{2}$ (1.3)

---

2. Kolmnurk horisontaalse alusega.



Raskuskese asub mediaanide lõikepunktis.

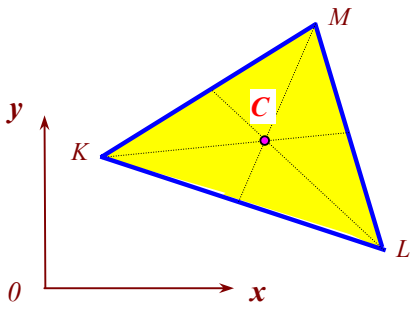
$$y_C = \frac{h_a}{3} \tag{2.1}$$

$$x_C = \frac{1}{3}(x_O + x_M + x_N), \tag{2.2}$$

Pindala: $A = \frac{ah_a}{2}$ (2.3)

---

3. Suvaline kolmnurk.



$$x_C = \frac{1}{3}(x_K + x_L + x_M) \tag{3.1}$$

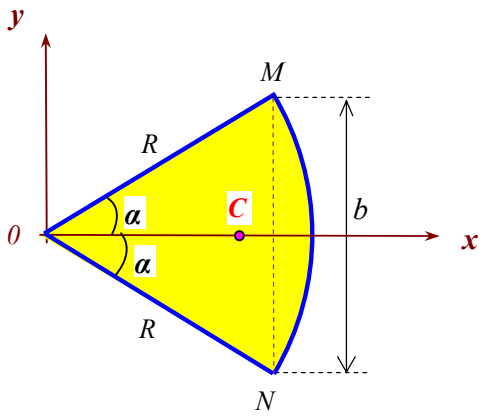
$$y_C = \frac{1}{3}(y_K + y_L + y_M) \tag{3.2}$$

Pindala:

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_K & y_K \\ 1 & x_L & y_L \\ 1 & x_M & y_M \end{vmatrix} \tag{3.3}$$

4. Ringi sektor.

4a) üldjuhtum:

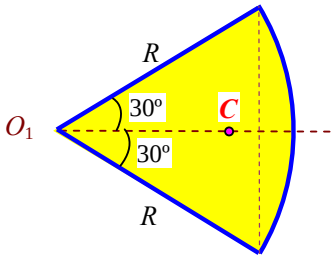


$$OC = \frac{2R\sin\alpha}{3\alpha} = \frac{R^2 b}{3A} \qquad (4.1)$$

Pindala: $A = \alpha R^2$ (4.2)

kus nurk α on ***radiaanides****.*

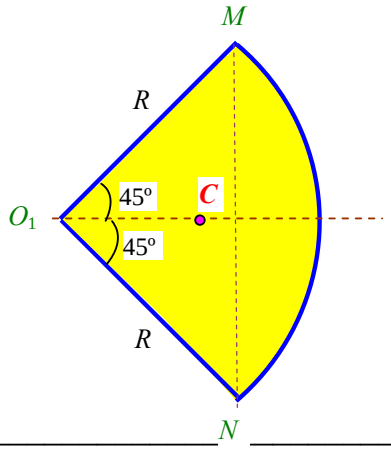
4b) nurk ***α*** on **30°**:



$$O_1C = \frac{2R}{\pi} \qquad (4.3)$$

Pindala: $A = \frac{\pi R^2}{6}$ (4.4)

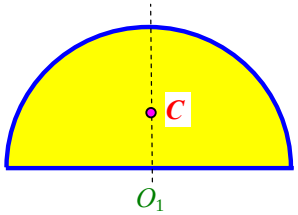
4c) nurk ***α*** on **45°**, sirged *OM* ja *ON* on risti:



$$O_1C = \frac{4\sqrt{2}\,R}{3\pi} \qquad (4.5)$$

Pindala: $A = \frac{\pi R^2}{4}$ (4.6)

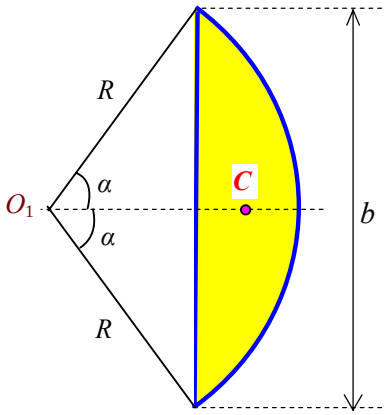
4d) nurk ***α*** on **90°**, kujund on poolring:



$$O_1C = \frac{4R}{3\pi} \qquad (4.7)$$

Pindala: $A = \frac{\pi R^2}{2}$ (4.8)

5. Ringi segment:



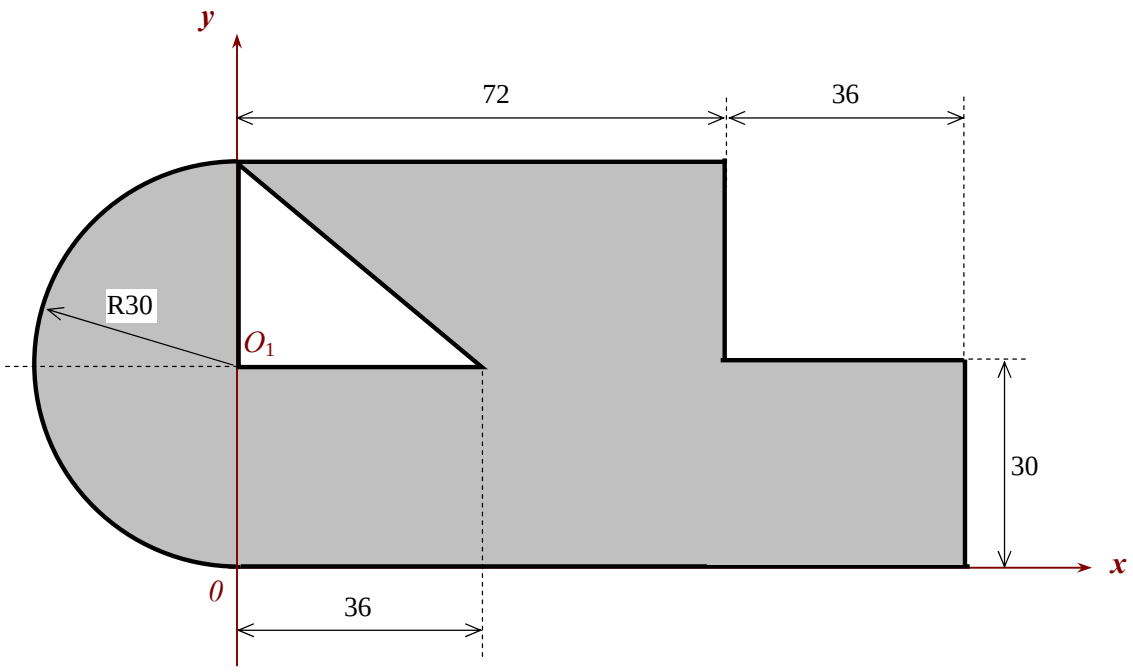
$$O_1C = \frac{2R\sin^3\alpha}{3(\alpha - \sin\alpha\cdot\cos\alpha)} = \frac{b^3}{12A} \qquad (5.1)$$

Pindala: $A = R^2(\alpha - \sin\alpha\cdot\cos\alpha)$ (5.2)

kus nurk α on ***radiaanides***.

Näiteülesanne nr 1

Leida joonisel 1 toodud kujundi raskuskese.



Joonis 1

Tasapinnalise kujundi raskuskeskme leidmine algab alati kujundi jaotamisest osadeks. Seda tuleb teha nii, et võetud osad oleksid kas:

1) ristkülik,
2) kolmnurk (soovitav täisnurkne kolmnurk),
3) ring või poolring,
4) ringi sektor,
5) ringi segment.

Tihti võib seda jaotamist teha mitut moodi, sel juhul valime neist kõige lihtsama. Kui me ei oska öelda, milline on kõige lihtsam, siis valime neist lihtsalt ühe välja. Nummerdame need osad ja leiame siis iga osa raskuskeskme. Need tähistame vastavalt osa numbrile C_1, C_2, C_3, … Paneme kirja iga osa raskuskeskme koordinaadid ja leiame iga osa pindala. Tulemused on mugav esitada tabeli kujul:

Tabel 1

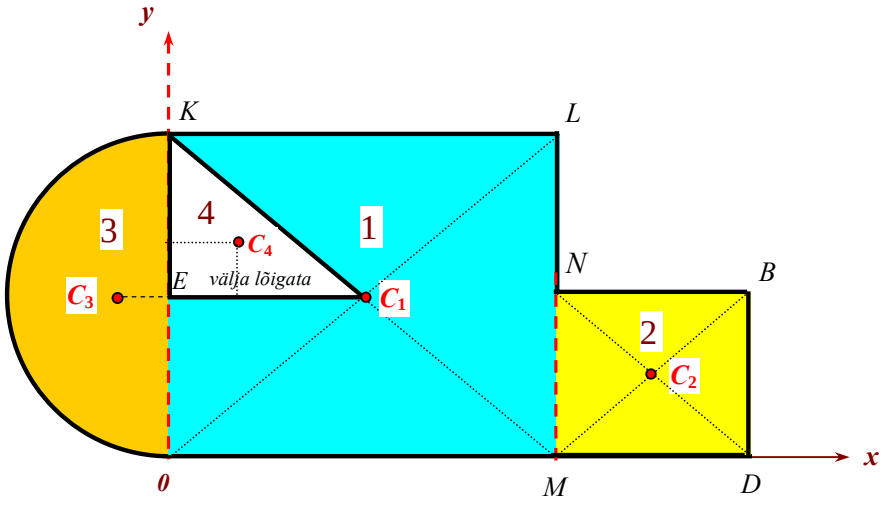
Osa nr. i	x_i	y_i	A_i	***Märkused*:**
1				
2				
…				
…				

Kogu kujundi raskuskeskme C leiame seejärel järgmiste valemitega:

$$x_C = \frac{x_1A_1 + x_2A_2 + \ldots + x_nA_n}{A_1 + A_2 + \ldots + A_n}\,; \qquad y_C = \frac{y_1A_1 + y_2A_2 + \ldots + y_nA_n}{A_1 + A_2 + \ldots + A_n} \tag{6}$$

Kui mingi osa on vaja välja lõigata, siis tuleb selle osa pindala võtta miinus märgiga. Seda võib realiseerida kahel viisil**: 1**) *miinusmärgi võib panna kohe tabelisse* 1; *sel juhul valemites* (6) *on kõik märgid plussid, sest miinus on juba vastava pindala* A_i *sees*; **2**) *panna tabelisse* 1 *kõik pindalad ikka positiivsed, aga miinuse panna valemitesse* (6) *vastava pindala* ***ette***. Võimaluse 1 kasutamist nimetatakse ***negatiivse pindala meetodiks.***

Jaotame joonisel 1 antud kujundi osadeks.



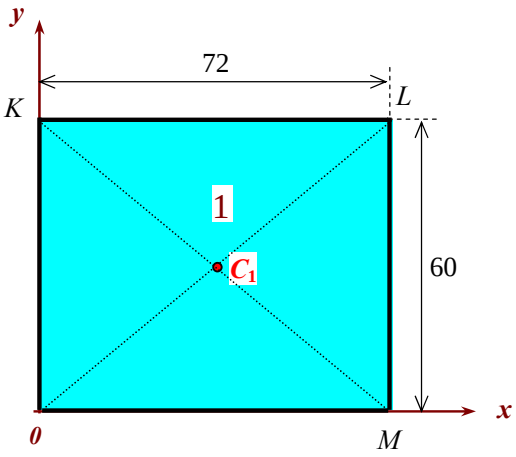
Joonis 2

Saime 4 osa (joonis 2), need on järgmised:

osa 1: suur ristkülik *0KLM*, selle raskuskese asub diagonaalide lõikepunktis C_1.
osa 2: väike ristkülik *MNBD*, selle raskuskese asub diagonaalide lõikepunktis C_2.
osa 3: poolring, mille raskuskeskme C_3 leiame valemi (4.7) abil.
osa 4: kolmnurk KEC_1, mis tuleb välja lõigata, selle raskuskeskme leiame valemite (1.1) ja (1.2) abil.

Vaatame igat osa eraldi.

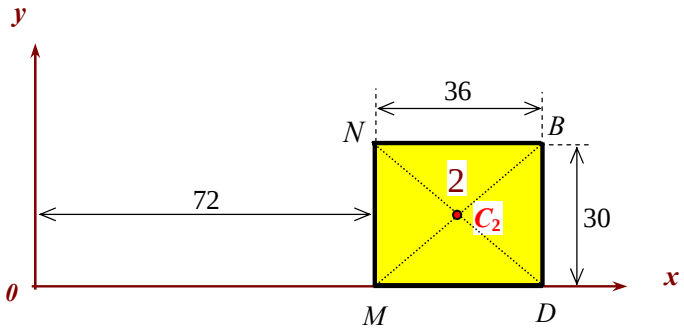
Osa 1.



Joonis 3a

$$x_1 = 36$$
$$y_1 = 30$$
$$A_1 = 72 \cdot 60 = 4320$$

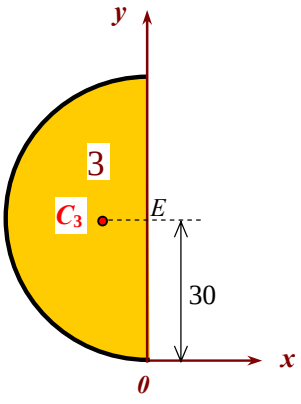
Osa 2.



Joonis 3b

$$x_2 = 72 + \frac{36}{2} = 90$$
$$y_2 = 15$$
$$A_2 = 36 \cdot 30 = 1080$$

Osa 3.



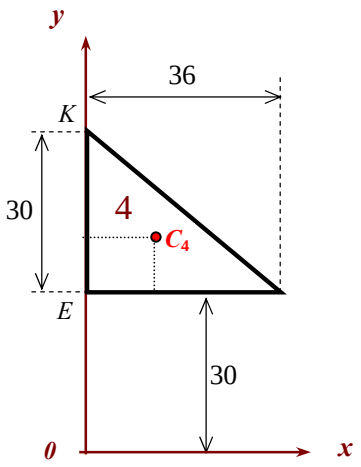
Joonis 3c

Valemi (4.7) alusel

$$EC_3 = \frac{4 \cdot R}{3 \cdot \pi} = \frac{4 \cdot 30}{3 \cdot 3{,}1416} = 12{,}732\,,$$ seetõttu

$$x_3 = -12{,}732$$
$$y_3 = 30$$
$$A_3 = \frac{\pi R^2}{2} = \frac{3{,}1416 \cdot 900}{2} = 1413{,}720$$

Osa 4.



Joonis 3d

$$x_4 = \frac{36}{3} = 12$$

$$y_4 = 30 + \frac{30}{3} = 40$$

Kasutades negatiivse pindala meetodit

$$A_4 = -\frac{30 \cdot 36}{2} = -540$$

Tulemused esitame tabeli kujul:

Tabel 2

Osa nr. *i*	x_i	y_i	A_i	***Märkused*:**
1	36	30	**4320**	Nelinurk *0KLM*
2	90	15	**1080**	Nelinurk *MNBD*
3	-12,732	30	**1413,72**	Poolring
4	12	40	**-540**	Kolmnurk, ***välja lõigata***

Seega kogu kujundi raskuskeskme koordinaadid tulevad järgmised

$$x_C = \frac{36 \cdot 4320 + 90 \cdot 1080 + (-12{,}732) \cdot 1413{,}72 + 12 \cdot (-540)}{4320 + 1080 + 1413{,}72 - 540} = \frac{228240{,}52}{6273{,}72} = 36{,}380$$

ja

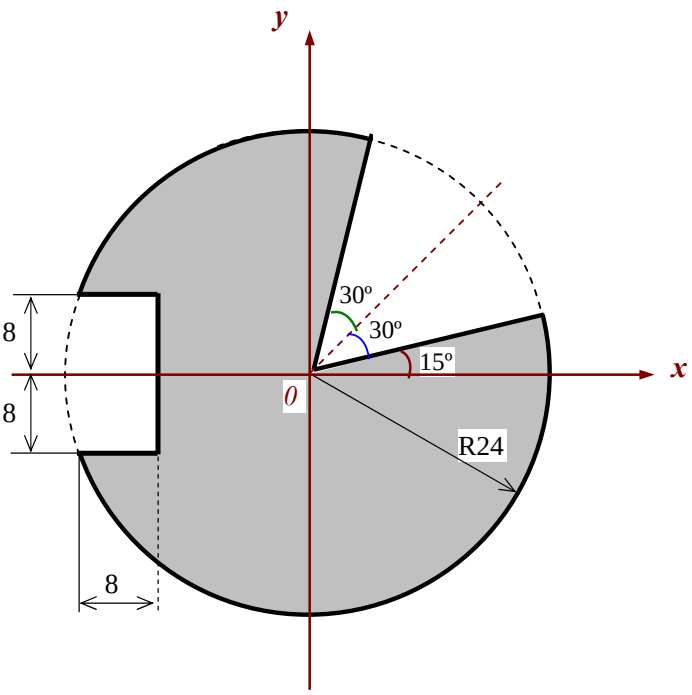
$$y_C = \frac{30 \cdot 4320 + 15 \cdot 1080 + 30 \cdot 1413{,}72 + 40 \cdot (-540)}{4320 + 1080 + 1413{,}72 - 540} = \frac{166611{,}6}{6273{,}72} = 26{,}557$$

Vastuse tuleb esitada täpsusega 3 kohta pärast koma.

Vastus: kujundi raskuskeskme koordinaadid on: $x_C = 36{,}380$, $y_C = 26{,}557$

Näiteülesanne nr 2

Leida joonisel 4 toodud kujundi raskuskese.



Joonis 4

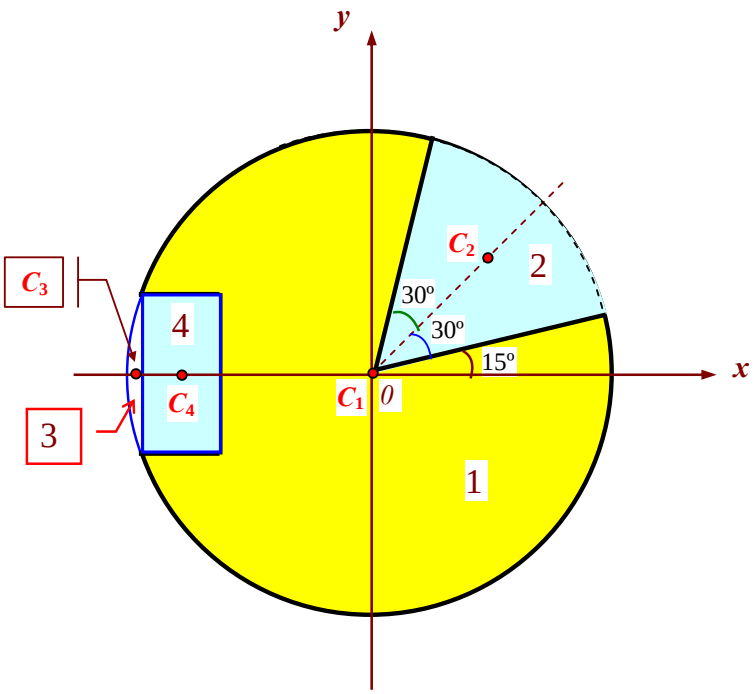
Jaotame kujundi kõigepealt osadeks. Nagu joonisel 4 on näha, on kujundi põhiosa ringjoon raadiusega 24, millest tuleb mitu osa välja lõigata. Väljalõigatavateks osadeks on: ristkülik, segment ja sektor. Joonisel **5** on näidatud selle kujundi kõik osad.

Osa 1: ringjoon raadiusega 24 ja raskuskeskmega punktis C_1 ehk *0*.

Osa 2: ringi sektor, raadiusega samuti 24 ja raskuskeskmega punktis C_2.

Osa 3: ringi segment raskuskeskmega punktis C_3.

Osa 4: ristkülik raskuskeskmega punktis C_4.



Joonis 5

Vaatame neid osasid lähemalt.

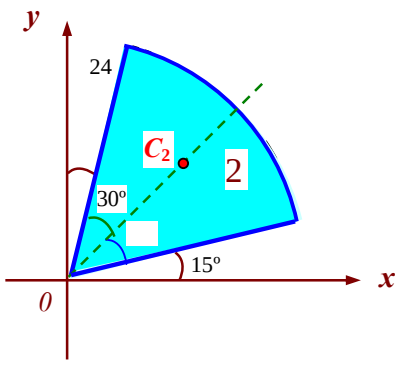
Osa 1.

Kuna ringjoone keskpunkt on ühtlasi koordinaatide alguspunktiks, siis

$$x_C = 0, \quad y_C = 0$$

Ringi pindala on $A_1 = \pi R^2 = 3{,}1416 \cdot 24^2 = 1809{,}5616$

Osa 2.



Joonis 6a

Väljalõigatav sektor, millel raadiuseks on 24 ja tipunurk on $2 \times 30° = 60°$. Selle raskuskeskme C_2 asukoha sektori sümmeetriateljel leiame valemi (4.3) abil, mille põhjal

$$OC_2 = \frac{2R}{\pi} = \frac{2 \cdot 24}{3{,}1416} = 15{,}2788$$

Seetõttu

$$x_2 = OC_2 \cdot \cos 45° = 10{,}8037$$

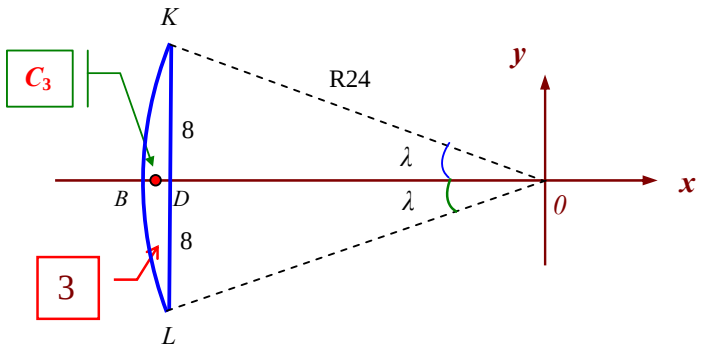
$$y_2 = OC_2 \cdot \cos 45° = 10{,}8037$$

Pindala

$$A_2 = \frac{\pi R^2}{6} = \frac{3{,}1416 \cdot 24^2}{6} = 301{,}5936$$

mille võtame miinusmärgiga, kuna osa lõigatakse välja.

Osa 3.



Joonis 6b

Väljalõigatav segment, mille kaareosa raadiuseks on 24 ja tipunurk on 2λ. Selle segmendi raskuskeskme C_3 asukoha määrame valemi (5.1) abil ja pindala A_3 valemiga (5.2). Nende alusel

$$|OC_3| = \frac{|KL|^3}{12\,A_3} = \frac{16^3}{12\,A_3}$$

$$A_3 = R^2\left(\lambda - \sin\lambda\cdot\cos\lambda\right) = 24^2\left(\lambda - \sin\lambda\cdot\cos\lambda\right)$$

Leiame nurga λ. Joonise 6b põhjal

$$\sin\lambda = \frac{|KD|}{R} = \frac{8}{24} = \frac{1}{3}$$

Seetõttu $\lambda = \arcsin 0{,}333333 = 0{,}3398369$ radiaani.

Siis

$$\cos\lambda = \cos 0{,}3398369 = 0{,}9428090$$

Seega pindala

$$A_3 = 24^2\left(0{,}3398369 - 0{,}3333333\cdot 0{,}942809\right) = 14{,}72676 \approx 14{,}7268$$

mille võtame miinusmärgiga, kuna osa on vaja välja lõigata.

Seega $$|OC_3| = \frac{16^3}{12\cdot 14{,}7268} = 23{,}1777$$

See tähendab, et

$$x_3 = -23{,}1777\,;\quad y_3 = 0$$

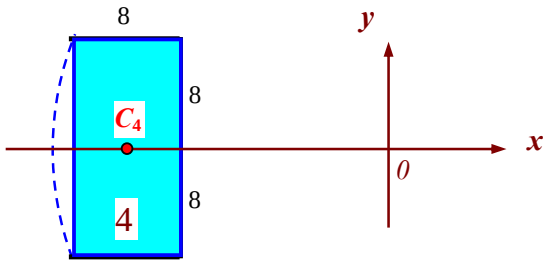
Leiame veel segmendi kõrguse BD, mis on vajalik ristküliku 4 raskuskeskme asukoha määramiseks. Tähistades selle h_3, saame joonise 6b põhjal

$$h_3 = |BD| = |OB| - |OD| = R - R\cos\lambda = R\left(1 - \cos\lambda\right)$$

ehk

$$h_3 = 24\left(1 - 0{,}942809\right) = 1{,}37258 \approx 1{,}3726$$

Osa 4.



Joonis 6c

Osaks 4 on **väljalõigatav** ristkülik (joonis 6c). Selle pindala on

$$A_4 = 16 \cdot 8 = 128$$

mille võtame samuti miinusmärgiga, kuna ka selle osa lõikame välja.

Osa 4 raskuskeskme koordinaat y_4 on muidugi 0, koordinaadi x_4 arvutame aga järgmiselt

$$x_4 = -\left(R - h_3 - \frac{8}{2} \right) = -\left(24 - 1{,}3726 - 4\right) = -18{,}6274$$

Tulemused esitame tabeli kujul:

Tabel 2

Osa nr. i	x_i	y_i	A_i	***Märkused*:**
1	0	0	**1809,5616**	Ring
2	10,8037	10,8037	**-301,5936**	Ringi sektor, ***välja lõigata***
3	-23,1777	0	**-14,7268**	Ringi segment, ***välja lõigata***
4	-18,6274	0	**-128**	Ristkülik, ***välja lõigata***

Kogu kujundi raskuskeskme koordinaadid tulevad valemite (6) põhjal järgmised

$$x_C = \frac{0 \cdot 1809{,}5616 + 10{,}8037 \cdot \left(-301{,}5936\right) + \left(-23{,}1777\right) \cdot \left(-14{,}7268\right) + \left(-18{,}6274\right) \cdot \left(-128\right)}{1809{,}5616 - 301{,}5936 - 14{,}7268 - 128}$$

millest

$$x_C = \frac{-532{,}6862}{1365{,}2412} = -0{,}390177 \approx -0{,}390$$

Analoogiliselt

$$y_C = \frac{0 \cdot 1809{,}5616 + 10{,}8037 \cdot \left(-301{,}5936\right) + 0 \cdot \left(-14{,}7268\right) + 0 \cdot \left(-128\right)}{1809{,}5616 - 301{,}5936 - 14{,}7268 - 128}$$

millest

$$y_C = \frac{-3258{,}3267}{1365{,}2412} = -2{,}38663 \approx -2{,}387$$

Vastus: kujundi raskuskeskme koordinaadid on: $\boldsymbol{x_C = -0{,}390}$, $\boldsymbol{y_C = -2{,}387}$