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CONTROL
SYSTEMS

Connections in control strategy

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Abstract. We present an infinitesimal interpretation of the control theory, particularly of the part concerning dynamic systems. We use the original concept of a bundle connection, which lies in the idea of fibre transportation along a path on the base manifold. The control of a process leads also to the transportation of fibres, and the control strategy, i.e. the choice of a suitable system control in order to optimize the process corresponds to the choice of a path on the base manifold. The triple of crucial terms of control, aim–control–strategy, translates in the terms of connections as fibre–connection–curve. Such a scheme is quite convincing, but it also works well in dynamic systems analysis.

Key words: control theory, connection.

1. INTRODUCTION

When controlling a system, we not only apply one control model but also try to find a more suitable control model among the possible ones, i.e. we search the control strategy. We distinguish the following stages: controlled process – control correction – strategy choice.

Let us describe the mathematical setting. Let X, Y, and Z be three vector fields and let us denote by

$$a_t = \exp tX, \quad b_\sigma = \exp \sigma Y, \quad c_\tau = \exp \tau Z$$

the appropriate flows. If we understand the flow as a motion, the vector field can be seen as stopping the motion at the precise moment (stop-scene). Shortly, a vector field is an infinitesimal representation of the flow.

The flow c_τ of the vector field Z represents the controlled process (it is also possible to replace it by a transport of an arbitrary tensor field along the flow c_τ). Furthermore, the flow b_σ acts on the vector field Z flow c_τ as follows; see [1,3]:

$$c_\tau \rightsquigarrow b_\sigma c_\tau b_\sigma^{-1}, \quad Z \rightsquigarrow Z_\sigma .$$

This corresponds to the change of control. If in addition the flow a_t acts on b_σ, we have a control strategy

$$b_\sigma \rightsquigarrow a_t b_\sigma a_t^{-1}, \quad Y \rightsquigarrow Y_t .$$

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Concerning the strategy $\{X \rightsquigarrow \{Y \rightsquigarrow Z\}\}$, it is obtained as a composition of the action of b_σ on c_τ and the action of a_t on b_σ,

$$c_\tau \rightsquigarrow b_\sigma c_\tau b_\sigma^{-1} \rightsquigarrow (a_t b_\sigma a_t^{-1}) c_\tau (a_t b_\sigma a_t^{-1})^{-1}.$$

The vector field Y plays the role of the one controlled by the vector field X and the role of the controlling field over Z.

The main goal of the paper is to describe the transformation of the parameters when the process Z is changed according to the strategy appropriate to the vector field X under the action of the vector field Y.

2. VECTOR FIELDS

Let M be a smooth manifold. The derivatives of a function $f : M \to \mathbb{R}$ along the vector fields X, Y, and Z are defined by

$$Xf \doteq (f \circ a_t)'_{t=0}, \quad Yf \doteq (f \circ b_\sigma)'_{\sigma=0}, \quad Zf \doteq (f \circ c_\tau)'_{\tau=0}.$$

The vector field Y is transported along the flow of X, which can be understood as an infinitesimal interpretation of such transportation (stop-scene) – the bracket of vector fields, i.e. Lie derivative $\mathscr{L}_X Y = [X,Y]$.

Remark 1. One can obtain the bracket of two vector fields $[X,Y] = XY - YX$ by double differentiation of a function f along the vector field flow $a_t b_\sigma a_t^{-1}$ w.r.t. σ and then w.r.t. t:

$$f \circ (a_t b_\sigma a_t^{-1})^{-1} \overset{(\cdot)'_{\sigma=0}}{\longrightarrow} -\big(Y(f \circ a_t)\big) \circ a_t^{-1} \overset{(\cdot)'_{t=0}}{\longrightarrow} (XY - YX)f.$$

Next, the transport of an arbitrary smooth tensor field along a vector field flow is defined by the Lie–Maclaurin series. For example, the transport of a vector field Z along the flow b_σ is defined by

$$Z \rightsquigarrow Z_\sigma = Z + Z'\sigma + Z''\frac{\sigma^2}{2} + \ldots = \sum_{k=0}^{\infty} Z^{(k)} \frac{\sigma^{(k)}}{k!},$$

where the coefficients $Z^{(k)} = \mathscr{L}_Y Z^{(k-1)}$, $k = 1, 2, \ldots$, are Lie derivatives of Z with respect to Y.

In our situation, the vector field X plays three roles:

1. The vector field X itself causes the process as a motion in its flow a_t.
2. The operator $\mathscr{L}_X$ transforms the control $Y \rightsquigarrow [X,Y]$.
3. The operator $\mathscr{L}_{\mathscr{L}_X}$ defines the control strategy – control of control $\mathscr{L}_Y \rightsquigarrow [\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$. Note that the above equality can be obtained from the Jacobi identity:

$$\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] = 0$$

or equivalently

$$\big[[X,Y],Z\big] = \big[X,[Y,Z]\big] - \big[Y,[X,Z]\big].$$

Note that the process can be influenced only by some outer process, not by it itself. Indeed, if we admit that the operator $\mathscr{L}_X$ acts on the control by the process X, we obtain:

$$\mathscr{L}_X X = [X,X] = 0.$$

We assign the following operators to the vector fields X, Y, and Z:

1. the operator Z implementing the process (motion in the flow c_τ);
2. the operator $\mathscr{L}_Y$ implementing the control of the process $Z \rightsquigarrow [Y,Z]$ (motion of the flow c_τ in the flow b_σ);
3. the operator $\mathscr{L}_{\mathscr{L}_X}$ defining the control strategy $\mathscr{L}_Y$ (motion in the flow a_t of a motion of the flow c_τ in the flow b_σ).

3. CONTROL AND CONNECTION

We will follow the notations appropriate to the theory of connections on fibred manifolds; see, e.g., [1,4].

Let us consider a vector bundle $\pi : M_1 \to M$ with n-dimensional base manifold M and r-dimensional fibres. The standard fibre is isomorphic to $\mathbb{R}^r$. On a neighbourhood $U \subset M_1$ we have local coordinates (u^i, u^α), where (u^i) denotes the base coordinates and (u^α) the fibre coordinates. Precisely, $u^i = \bar{u}^i \circ \pi$, where $\bar{u}^i$ denotes the local coordinates on the neighbourhood $\pi(U) \subset M$. The coordinates (u^α) are the coordinates of $\mathbb{R}^r$. Latin indices $i, j, \ldots$ range from 1 to n, Greek indices $\alpha, \beta, \ldots$ range from $n+1$ to $n+r$.

We define two vector fields:

$$Y = y^\alpha \partial_\alpha \quad \text{and} \quad Z = z^\alpha \partial_\alpha .$$

Here z^α are the functions depending on the fibre coordinates u^α only, while y^α are the functions of all coordinates (u^i, u^α). The flow $b_\sigma = \exp \sigma Y$ is defined on the neighbourhood U by a system of ODEs

$$\frac{du^\alpha}{d\sigma} = y^\alpha(u^i, u^\beta). \tag{1}$$

Indeed, now we can see the connection between dynamic systems, see [2], and the controlling parameters (u^i). As mentioned above, these parameters are lifted from the base $\pi(U) \subset M$ to the neighbourhood $U \subset M_1$, i.e. $u^i = \bar{u}^i \circ \pi$.

On every fibre, the vector field Y induces a family of trajectories – phase portrait. When the fibre is changed, the vector field Y changes too and so does the phase portrait, i.e. the control $\{Y \rightsquigarrow Z\}$. A question arises: how do the parameters (u^i) affect the controlling process?

Let us consider the coordinate map

$$\Phi : (u^i, u^\alpha) \rightsquigarrow (u^i, s, I^\kappa),\ k = n+2, \ldots, n+r,$$

where s is a canonical parameter, i.e. $\mathscr{L}_Y s = 0$, and I^κ is a system of $r-1$ independent invariants of the vector field Y. The coordinates (u^i, I^κ) form a complete system of local invariants of Y on the manifold M_1. Now we can define the submersion of the manifold M_1 onto the fibre $\mathbb{R}^r$,

$$\varphi : M_1 \to \mathbb{R}^r : (u^i, u^\alpha) \rightsquigarrow (s, I^\kappa).$$

A fibre of the submersion φ has the dimension n and forms the family of the integral surfaces which define a horizontal distribution $\triangle_h$. Thus on the fibration π, a zero torsion connection structure $\triangle_h \oplus \triangle_v$ is defined. Let us consider the *adapted basis*

$$(X_i\, X_\alpha) = \begin{pmatrix} \dfrac{\partial}{\partial u^j} & \dfrac{\partial}{\partial u^\beta} \end{pmatrix} \cdot \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i^\beta & \delta_\alpha^\beta \end{pmatrix}, \quad \begin{pmatrix} \omega^i \\ \omega^\alpha \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ -\Gamma_j^\alpha & \delta_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} du^j \\ du^\beta \end{pmatrix},$$

where the vector fields

$$X_i = \frac{\partial}{\partial u^i} + \Gamma_i^\alpha \frac{\partial}{\partial u^\alpha}$$

form a base of the distribution Δ_h and the forms

$$\omega^\alpha = du^\alpha - \Gamma_i^\alpha du^i$$

vanish on the distribution Δ_h. The number of parameters Γ_i^α equals nr and they define the distribution Δ_h uniquely. On the other hand, the parameters Γ_i^α are determined by setting the functions φ^α equal to a constant on the fibres of the submersion φ, more precisely by their differentials:

$$d\varphi^\alpha = \varphi_i^\alpha du^i + \varphi_\beta^\alpha du^\beta = \varphi_\beta^\alpha (du^\beta + \bar{\varphi}_\gamma^\beta \varphi_i^\gamma du^i) = \varphi_\beta^\alpha \omega^\beta \quad \Longrightarrow \quad \Gamma_i^\alpha = \bar{\varphi}_\gamma^\alpha \varphi_i^\gamma,$$

where the coefficients of $d\varphi^\alpha$ are the partial derivatives of φ^α. The matrix (φ_β^α) is the integrating matrix with respect to the forms ω^α and its inverse is $(\bar{\varphi}_\alpha^\beta)$.

Theorem 1. *The vector field Y is projected by the submersion $\varphi : M_1 \to \mathbb{R}^r$ onto the vector field $T\varphi Y$ on the standard fibre $\mathbb{R}^r$. In the coordinates (s,I), where s denotes the canonical parameter and I is a system of the base invariants, the vector field $T\varphi Y$ represents the operator $\partial_s \doteq \frac{\partial}{\partial s}$. The vector field Z is expressed uniquely in the basis (∂_s, ∂_I) and the process controlled by Z is, in the coordinate system (s,I), described by the functions $s \circ \varphi$ and $I \circ \varphi$. These functions depend on the parameters u^α and the controlling parameters u^i.*

Proof. A family of the fibres corresponding to the submersion φ is defined by the solution of the system of differential equations $(u^\alpha)_\sigma = \varphi^\alpha(\sigma, u^i, u^\beta)$; see system (1). Furthermore, an arbitrary section of the fibration π can be extended into the system of imprimitivity appropriate to the flow b_s, i.e. the family of the fibres corresponding to the submersion φ. The vector field Y is φ-projected on the fibre $\mathbb{R}^r$. An integrable distribution $\Delta_h = \mathrm{Ker} T\varphi$ in the fibration π defines a zero curvature connection and thus on the neighbourhood U the basis and the co-basis of the distribution Δ_h is defined as follows:

$$X_i = \partial_i + \Gamma_i^\alpha \partial_\alpha, \quad \omega^\alpha = du^\alpha - \Gamma_i^\alpha du^i.$$

Let us recall that an arbitrary vector field $\bar{X}$ on the base manifold M can be lifted from M to the horizontal distribution Δ_h :

$$\bar{X} = \bar{x}^i \bar{\partial}_i \quad \rightsquigarrow \quad X = x^i X_i, \text{ where } x^i = \bar{x}^i \circ \varphi.$$

In our notations, the basis X_i represents the operators $\bar{\partial}_i$ from the neighbourhood $\pi(U)$ lifted to the distribution Δ_h.

It is now clear that the vector field X behaves with respect to the vector field Y as an infinitesimal symmetry, i.e. $[X,Y] = 0$, and thus the impact on the vector field Y vanishes. In other words, the process appropriate to the vector field Z is defined on the fibre in the coordinates (s,I), where the functions $s \circ \varphi$ and $I \circ \varphi$ depend on the parameters u^α and the controlling parameters u^i. The vector field X affects the vector field Z indirectly by means of the invariants of the vector field Y. □

Remark 2. The components y^α of the vector field Y depend linearly and homogeneously on the fibre coordinates. Thus the defining system is described by the system of linear differential equations

$$\frac{du^\alpha}{d\sigma} = y^\alpha_\beta(u^i) u^\beta.$$

4. APPLICATION

On the bundle[1]

$$\pi : \mathbb{R}^3 \to \mathbb{R} : (u,x,y) \rightsquigarrow (u)$$

with the fibre coordinates (x,y) and the controlling parameter (or base coordinate) (u) we have the vector field

$$Y = \frac{\partial}{\partial x} + ux\frac{\partial}{\partial y}.$$

We define its flow $b_s = \exp sY$, the canonical parameter s, and the invariant I of Y as follows:

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = ux \end{cases} \Rightarrow \begin{cases} x_s = x + s \\ y_s = y + u(xs + \frac{s^2}{2}), \end{cases} \qquad \begin{cases} s = x \\ I = y - \frac{ux^2}{2}. \end{cases}$$

We check that $\mathscr{L}_Y s = 1, \mathscr{L}_Y I = 0$. The trajectories on the fibres are parabolas depending on the parameter u.

[1] Here, for the sake of simplicity, we denote the local coordinates by (u,x,y) instead of (u^1,u^2,u^3) but note that the fibre coordinates (x,y) are in no way related to the components (x^i, y^α) of the vector fields X and Y.

The submersion $\varphi : \mathbb{R}^3 \to \mathbb{R} : (u,x,y) \rightsquigarrow (s,I)$ projects the space $\mathbb{R}^3$ onto the plane sI. The tangent mapping $T\varphi$ is defined by the following differentials and by the Jacobi matrix:

$$\begin{cases} ds = dx \\ dI = -\frac{x^2}{2}du - uxdx + dy, \end{cases} \qquad \begin{pmatrix} 0 & 1 & 0 \\ -\frac{x^2}{2} & -ux & 1 \end{pmatrix}.$$

The vector field Y with the components $(0,1,ux)$ is projected to the plane sI in which it forms the operator $T\varphi Y = \partial_s$ (see Fig. 1).

Thus on the bundle π a horizontal distribution

$$\triangle_h = \operatorname{Ker} T\varphi$$

is defined. The co-basis on $\triangle_h$ is of the form

$$\begin{cases} \omega^2 = \quad ds \quad = dx \quad = (dx - \Gamma_1^2 du) \\ \omega^3 = \quad uxds + dI = dy - \frac{x^2}{2}du = (dy - \Gamma_1^3 du), \end{cases}$$

and the connection coefficients are

$$\begin{pmatrix} \Gamma_1^2 \\ \Gamma_1^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{x^2}{2} \end{pmatrix}.$$

The adapted basis of the distribution $\triangle_h$ is characterized by the following:

$$X_1 = \partial_u + \frac{x^2}{2}\partial_y, \qquad \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{x^2}{2} \end{pmatrix} \cdot (du).$$

The operator X_1 commutes with the vector field Y, i.e. $[X_1, Y] = 0$, and vanishes under the projection $T\varphi$, i.e. $T\varphi X_1 = 0$. The co-basis admits an integrating matrix as follows:

$$\begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ us & 1 \end{pmatrix} \cdot \begin{pmatrix} ds \\ dI \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ -ux & 1 \end{pmatrix} \cdot \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} ds \\ dI \end{pmatrix}.$$

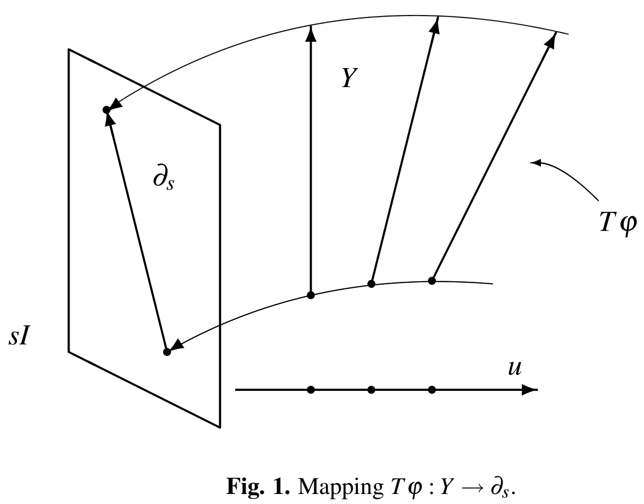

Fig. 1. Mapping $T\varphi : Y \to \partial_s$.

The direct impact of the parameter (u) on the operator Y is eliminated. Indeed, because the projection φ targets on the fibre xy, it is possible to change the coordinates under the condition $u = \text{const}$ from (x,y) to (s,I),

$$\begin{cases} x = s \\ y = \frac{us^2}{2} + I, \end{cases} \qquad \begin{pmatrix} 1 & 0 \\ us & 1 \end{pmatrix}.$$

Using the Jacobi matrix (the right-hand side), we can change the basis to the new natural one and we obtain the following frames and co-frames:

$$\left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial s}\ \frac{\partial}{\partial I}\right) \cdot \begin{pmatrix} 1 & 0 \\ -us & 1 \end{pmatrix}, \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ us & 1 \end{pmatrix} \cdot \begin{pmatrix} ds \\ dI \end{pmatrix}.$$

Let us focus on the fibre. Note that the action of the vertical vector field Z can be understood as an action on a tensor field. Concerning the action of the operator Y on the vector field Z in the form

$$Y \rightsquigarrow Z = \mu \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial y},$$

with the components (μ, ν), we can see that in new coordinates it reduces to the action of the operator ∂_s on the vector field $\tilde{Z}$ depending on the parameter u only:

$$\partial_s \rightsquigarrow \tilde{Z} = \mu \partial_s + \nu \partial_I - u\mu s\, \partial_I.$$

Note that Z and $\tilde{Z}$ are the same vector field, only expressed in the coordinates (x,y) and (s,I), respectively. The operators $T\varphi Y$ and ∂_s are the same operators expressed in different coordinate systems.

Thus we change the control:

$$\{Y \rightsquigarrow Z\} \quad \rightsquigarrow \quad \{\partial_s \rightsquigarrow \tilde{Z}\}.$$

Remark 3. As an example, let us consider the operator of rotation

$$Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

In coordinates (s,I), it can be written in the form $\tilde{Z} = -I\partial_s + s\partial_I + u(\ldots)$, i.e. in such a form that some new operator with coefficient u is added. Such a property holds for an arbitrary linear dynamic system.

The control $\{Y \rightsquigarrow Z\}$ is described in the coordinates (x,y), while the control $\{\partial_s \rightsquigarrow \tilde{Z}\}$ is expressed in the coordinates (s,I). The parameter u affects the controlled field $\tilde{Z}$ directly.

5. CONCLUSION

The control of a dynamic system is viewed by means of differential geometry as the vector field Y on the bundle $\pi : M_1 \to M$ with the standard fibre $\mathbb{R}^r$ and the base manifold $M = \mathbb{R}^n$. The submersion φ is defined in such a way that the vector field Y is projected to the fibre $\mathbb{R}^r$. The distribution $\triangle_h = \mathrm{Ker}T\varphi$ gives rise to the possibility of eliminating the dependence of the vector field Y on the controlling parameter u. The change of variables to (s,I), where s is the canonical parameter and I is the invariant of the field Y, changes the control $(Y \to Z)$ to the control $\{\partial_s \rightsquigarrow \tilde{Z}\}$, where the field ∂_s no longer depends on the parameter u while the controlled field $\tilde{Z}$ does so.

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Seostused juhtimise teoorias

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Juhtimisel on kolm aspekti: juhitav protsess, juhtiv protsess ja juhtimise valik/strateegia. Matemaatiliseks mudeliks on kihtkond, kus seostus mõjutab toimuvat kihil, ja baasiparameetrid, millest sõltub seostus. Need määravad juhtimise strateegia. Osutub, et baasiparameetreid võib seostuse abil otsekohe kihile suunata, st juhtivast protsessist juhitavasse.