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ALGEBRA

Quotients and subalgebras of sup-algebras

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Abstract. An ordered algebra is called a sup-algebra if its underlying poset is a complete lattice and its operations are compatible with joins in each variable. In this article we study quotients and subalgebras of sup-algebras. We show that the congruence lattice of a sup-algebra is isomorphic to the lattice of its nuclei and dually isomorphic to the lattice of its meet-closed subalgebras. We also prove that the lattice of subalgebras of a sup-algebra is isomorphic to the lattice of its conuclei.

Key words: sup-algebra, nucleus, conucleus, congruence, quotient, subalgebra.

1. PRELIMINARIES

Various quantale-like structures (quantales, locales, quantale modules, quantale algebras, unital quantales, etc.) have been studied for decades and they have useful applications in algebra, logic, and computer science ([7,8]). There are a number of results for which the proofs are very similar for different structures. This suggests that perhaps those results could be obtained in a uniform way that is similar to how universal algebra gives a framework for studying algebraic structures. This approach was used by Resende in [6], where the author calls a quantale-like analogue of a universal algebra a *sup-algebra*, and followed in [5]. Our aim in this text is to develop this nice idea further. Although in [6] many-sorted sup-algebras are considered, we restrict ourselves to the one-sorted case (as was also done in [5]).

In this paper we will study the relationships between nuclei and congruences and between conuclei and subalgebras of sup-algebras, using an approach that is similar to the one used in [10] for the case of quantale algebras. We will also generalize a well-known representation theorem of quantales to the sup-algebra setting. Our point of view is that a sup-algebra is an ordered algebra with a 'very good' order structure.

Definition 1 ([1])**.** *Let Ω be a type. An* ordered Ω-algebra (*or simply an ordered algebra*) *is a triplet $\mathscr{A} = (A, \Omega_A, \leqslant_A)$ comprising a poset $(A, \leqslant_A)$ and a set Ω_A of operations on A (for every k-ary operation symbol $\omega \in \Omega_k$ there is a k-ary operation $\omega_A \in \Omega_A$ on A) such that all the operations ω_A are monotone mappings, where monotonicity of ω_A ($\omega \in \Omega_k$) means that*

$$a_1 \leqslant_A a_1' \wedge \ldots \wedge a_k \leqslant_A a_k' \Longrightarrow \omega_A(a_1, \ldots, a_k) \leqslant_A \omega_A(a_1', \ldots, a_k')$$

for all $a_1, \ldots, a_k, a_1', \ldots, a_k' \in A$.

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Definition 2 (cf. Def. 2.2.1 of [6])**.** *Let Ω be a type and let $\mathscr{A} = (A, \Omega_A, \leqslant_A)$ be an ordered Ω-algebra. We say that $\mathscr{A}$ is a* sup-Ω-algebra *if the poset $(A, \leqslant)$ is a complete lattice and*

$$\omega_A(a_1, \ldots, a_{i-1}, \bigvee M, a_{i+1}, \ldots, a_n) = \bigvee \{\omega_A(a_1, \ldots, a_{i-1}, m, a_{i+1}, \ldots, a_n) \mid m \in M\}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$, and *$M \subseteq A$.*

Remark 3. For the whole paper, we fix a type Ω and instead of sup-Ω-algebras we simply speak about sup-algebras.

Definition 2 covers a large variety of quantale-like structures. For example, the following structures are sup-algebras for some specific type Ω:
(1) quantales,
(2) unital quantales ([7], Definition 2.1.4),
(3) prequantales ([7], Definition 2.4.2),
(4) locales (frames) ([7], Definition 1.2.1),
(5) quantale modules ([8], Definition 4.1.1),
(6) quantale algebras ([9]),
(7) involutive quantales ([4], Definition 1.1.9),
(8) continuous semirings that are complete lattices ([3], Definition 3),
(9) S-quantales, where S is a posemigroup or a pomonoid ([11], Definition 3).

2. HOMOMORPHISMS AND SUBHOMOMORPHISMS

Definition 4. *Let $\mathscr{A}$ and $\mathscr{B}$ be sup-algebras. A mapping $f : A \to B$ is a* homomorphism of sup-algebras *if f preserves all basic operations and joins.*

Remark 5. We point out that a homomorphism of sup-algebras has to preserve also nullary operations. Thus a homomorphism $f : \mathscr{A} \to \mathscr{B}$ of unital quantales has to preserve the identity element, $f(1_A) = 1_B$. Also, any homomorphism of sup-algebras preserves the smallest element of the lattice (as the join of the empty family).

Definition 6. *Let $\mathscr{A}, \mathscr{B}$ be ordered Ω-algebras. We say that a monotone mapping $f : A \to B$ is a* subhomomorphism *if*

$$\omega_B(f(a_1), \ldots, f(a_n)) \leqslant f(\omega_A(a_1, \ldots, a_n))$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A$, and

$$\omega_B \leqslant f(\omega_A)$$

for every $\omega \in \Omega_0$.

Remark 7. In the case when $\mathscr{A}, \mathscr{B}$ are quantales (unital quantales), a mapping $f : A \to B$ with such a property is called a closed map of quantales (respectively, a closed unital map of quantales; see Definition 2.3.2 in [7]).

From the definitions it follows that every homomorphism of sup-algebras is a subhomomorphism. The composite of two homomorphisms (subhomomorphisms) is a homomorphism (resp. subhomomorphism), and the identity transformation of a sup-algebra is both a homomorphism and a subhomomorphism. Hence one may consider the category of sup-algebras where morphisms are subhomomorphisms, or its subcategory where morphisms are homomorphisms. It is easy to see that the following result holds.

Proposition 8. *Isomorphisms in the category of sup-algebras and their homomorphisms are precisely surjective homomorphisms that are order-embeddings.*

In a standard way, every poset can be considered as a category, and monotone mappings between posets can be considered as functors. In such a category coproducts are joins. Since a sup-algebra homomorphism $f : \mathscr{A} \to \mathscr{B}$ preserves all joins, it follows by the adjoint functor theorem that it has a right adjoint $f_* : \mathscr{B} \to \mathscr{A}$. In particular,

$$f(a) \leqslant b \Longleftrightarrow a \leqslant f_*(b), \tag{2.1}$$

$$a \leqslant f_*(f(a)), \text{ and } f(f_*(b)) \leqslant b \tag{2.2}$$

for all $a \in A$, $b \in B$.

The following result is a generalization of Proposition 2.3.3 in [8].

Proposition 9. *Let $\mathscr{A}, \mathscr{B}$ be sup-algebras, and let $f : \mathscr{A} \to \mathscr{B}$ be a homomorphism of sup-algebras. Then $f_* : \mathscr{B} \to \mathscr{A}$ is a subhomomorphism.*

Proof. It is clear that f_* is order-preserving. To prove the inequality

$$\omega_A(f_*(b_1), \ldots, f_*(b_n)) \leqslant f_*(\omega_B(b_1, \ldots, b_n))$$

for $n \in \mathbb{N}$, $\omega \in \Omega_n$, $b_1, \ldots, b_n \in B$, due to (2.1) it is sufficient to show that the inequality $f(\omega_A(f_*(b_1), \ldots, f_*(b_n))) \leqslant \omega_B(b_1, \ldots, b_n)$ holds. Since f preserves operations, from (2.2) it follows that

$$f(\omega_A(f_*(b_1), \ldots, f_*(b_n))) = \omega_B(f(f_*(b_1)), \ldots, f(f_*(b_n))) \leqslant \omega_B(b_1, \ldots, b_n).$$

If $\omega \in \Omega_0$, then $f(\omega_A) = \omega_B$ implies that $\omega_A \leqslant f_*(\omega_B)$ by (2.1). □

3. NUCLEI

Nuclei play an important role in the theories of quantale-like structures. This section is dedicated to the basic properties of nuclei of sup-algebras.

Definition 10. (cf. [6], Def. 2.2.5). *A closure operator j on a sup-algebra $\mathscr{A}$ is a* nucleus *if it is a subendomorphism of $\mathscr{A}$.*

Each homomorphism of sup-algebras induces a nucleus.

Lemma 11 ([6], Proposition 2.2.9). *If $f : \mathscr{A} \to \mathscr{B}$ is a homomorphism of sup-algebras, then $j = f_* f$ is a nucleus on $\mathscr{A}$.*

The next result follows from Lemma 2.2.6 of [6].

Lemma 12. *If j is a nucleus on a sup-algebra $\mathscr{A}$, then*

$$j(\omega_A(a_1, \ldots, a_n)) = j(\omega_A(a_1, \ldots, a_{i-1}, j(a_i), a_{i+1}, \ldots, a_n))$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_n \in A$. Hence

$$j(\omega_A(a_1, \ldots, a_n)) = j(\omega_A(j(a_1), \ldots, j(a_n))).$$

We denote by $\mathsf{Nuc}(\mathscr{A})$ the set of all nuclei on a sup-algebra $\mathscr{A}$. Define a partial order on $\mathsf{Nuc}(\mathscr{A})$ by pointwise ordering. If j is a nucleus on $\mathscr{A}$, then we put

$$A_j := \{a \in A \mid j(a) = a\}.$$

By Proposition 2.2.8(4) in [6],

$$j \leqslant k \text{ in } \mathsf{Nuc}(\mathscr{A}) \Longleftrightarrow A_k \subseteq A_j.$$

The following result is a consequence of Proposition 2.2.8 in [6].

Proposition 13. *Let $\mathscr{A}$ be a sup-algebra. Then* $\mathsf{Nuc}(\mathscr{A})$ *is a complete lattice.*

Our next aim is to find out which subsets of A are equal to some subset A_j for a nucleus j on a sup-algebra $\mathscr{A}$.

Elementary translations on a sup-algebra $\mathscr{A}$ are the mappings of the form $\omega_i := \omega_A(a_1,\dots,a_{i-1},\_,a_{i+1},\dots,a_n) : A \to A$, where $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1,\dots,n\}$ and $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n$ are some fixed elements of A. Since an elementary translation ω_i preserves joins, it has a right adjoint, which we denote by $\omega_i^* : A \to A$, satisfying

$$\omega_i(a) \leqslant b \Longleftrightarrow a \leqslant \omega_i^*(b), \tag{3.1}$$

for all $a,b \in A$, and also

$$\omega_i(\omega_i^*(a)) \leqslant a \quad \text{and} \quad b \leqslant \omega_i^*(\omega_i(b)). \tag{3.2}$$

Lemma 14. *If j is a nucleus on a sup-algebra $\mathscr{A}$, then $j(\omega_i^*(b)) \leqslant \omega_i^*(j(b))$ for every $b \in A$.*

Proof. Let ω_i be as above. Due to (3.1), it is sufficient to show that $\omega_i(j(\omega_i^*(b))) \leqslant j(b)$. This holds because

$$\begin{aligned}
\omega_i(j(\omega_i^*(b))) &= \omega_A(a_1,\dots,a_{i-1},j(\omega_i^*(b)),a_{i+1},\dots,a_n)\\
&\leqslant \omega_A(j(a_1),\dots,j(a_{i-1}),j(\omega_i^*(b)),j(a_{i+1}),\dots,j(a_n))\\
&\leqslant j\,(\omega_A(a_1,\dots,a_{i-1},\omega_i^*(b),a_{i+1},\dots,a_n))\\
&= j(\omega_i(\omega_i^*(b)))\\
&\leqslant j(b).
\end{aligned}$$

□

The next result generalizes Proposition 3.1.2 of [7].

Proposition 15. *If $\mathscr{A}$ is a sup-algebra and $S \subseteq A$, then $S = A_j$ for some nucleus j on $\mathscr{A}$ if and only if S is closed under meets and under right adjoints of elementary translations.*

Proof. **Necessity.** Suppose that $S = A_j$ for a nucleus j on $\mathscr{A}$. It is easy to see that A_j is closed under meets. By Lemma 14, for $s \in S$, from

$$\omega_i^*(s) \leqslant j(\omega_i^*(s)) \leqslant \omega_i^*(j(s)) = \omega_i^*(s),$$

we conclude that $\omega_i^*(s) \in S$.

Sufficiency. Assume that S is closed under meets and under right adjoints of elementary translations. Define a mapping $j : A \to A$ by

$$j(a) := \bigwedge\{s \in S \mid a \leqslant s\} = \bigwedge(S \cap a\!\uparrow)$$

for $a \in A$. Then it is routine to check that j is a closure operator. It is clear that $S \subseteq A_j$. Since S is closed under meets, we also have $A_j \subseteq S$, and thus $S = A_j$. We show that j is a subhomomorphism. To prove the inequality $\omega(j(a_1),\dots,j(a_n)) \leqslant j(\omega(a_1,\dots,a_n))$ it suffices to show that $\omega(j(a_1),\dots,j(a_n)) \leqslant s$ whenever $s \in S$ and $\omega(a_1,\dots,a_n) \leqslant s$. Consider such s and denote $w_1 = \omega(\_,a_2,\dots,a_n)$. Then

$$\omega_1(a_1) \leqslant s \Longrightarrow a_1 \leqslant \omega_1^*(s) \in S \Longrightarrow j(a_1) \leqslant \omega_1^*(s) \Longrightarrow \omega_1(j(a_1)) \leqslant s.$$

Now let $\omega_2 = \omega(j(a_1),\_,a_3,\dots,a_n)$. Then

$$\omega_2(a_2) \leqslant s \Longrightarrow a_2 \leqslant \omega_2^*(s) \in S \Longrightarrow j(a_2) \leqslant \omega_2^*(s) \Longrightarrow \omega_2(j(a_2)) \leqslant s.$$

Continuing in this manner we arrive at $\omega(j(a_1),\dots,j(a_n)) \leqslant s$, which was needed.

Since j is increasing, $\omega_A \leqslant j(\omega_A)$ for every $\omega \in \Omega_0$. □

4. QUOTIENTS OF SUP-ALGEBRAS

There are two ways of forming quotients of quantale-like structures: using nuclei or congruences. We will study how these two approaches are interrelated in the context of sup-algebras.

For a sup-algebra $\mathscr{A}$ and a nucleus j on $\mathscr{A}$, the set A_j is a complete lattice with respect to the order inherited from A, where joins are given by

$$\bigvee^{j} M = j\left(\bigvee M\right)$$

for every $M \subseteq A_j$. By the construction, $A \to A_j, a \mapsto j(a)$ is a surjective mapping that preserves joins. On the set A_j we define operations by

$$\omega_{A_j}(a_1,\ldots,a_n) := j\left(\omega_A(a_1,\ldots,a_n)\right),$$

$n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1,\ldots,a_n \in A_j$, and by

$$\omega_{A_j} := j(\omega_A)$$

if $\omega \in \Omega_0$.

The next result is a corollary of Theorem 2.2.7 in [6].

Proposition 16. *For a sup-algebra $\mathscr{A}$ and a nucleus j on $\mathscr{A}$, A_j is also a sup-algebra with respect to the order and operations defined above.*

We denote the resulting sup-algebra by $\mathscr{A}_j$ and call it a *quantic quotient* of $\mathscr{A}$.

In [6], the author considers classes of sup-algebras that are defined using some set E of identities (equations) as generalizations of usual varieties (although the author does not use the word 'variety'). However, in the theory of ordered universal algebras, varieties are given by sets of inequalities (see, e.g. [1]).

An *inequality* of type Ω is a sequence of symbols $t \le t'$, where t, t' are Ω-terms. We say that $t \le t'$ holds in an ordered algebra $\mathscr{A}$ if $t_{\mathscr{A}} \le t'_{\mathscr{A}}$ where $t_{\mathscr{A}}, t'_{\mathscr{A}} : A^n \to A$ are the functions on $\mathscr{A}$ induced by t and t'. Of course, inequalities $t \le t'$ and $t' \le t$ hold in $\mathscr{A}$ if and only if the identity $t = t'$ holds in $\mathscr{A}$. By a *variety* of sup-algebras of type Ω we mean a class of sup-algebras satisfying some set E of inequalities and identities of type Ω. It turns out that such classes are closed under quantic quotients.

Proposition 17. *If $\mathscr{A}$ is a sup-algebra that belongs to some variety of sup-algebras, then $\mathscr{A}_j$ belongs to the same variety for every nucleus j on $\mathscr{A}$.*

Proof. We note that if $t = t(x_1,\ldots,x_n)$ and $t' = t'(x_1,\ldots,x_n)$ are two Ω-terms and $\mathscr{A}$ satisfies an inequality $t \leqslant t'$ then

$$t_{\mathscr{A}_j}(a_1,\ldots,a_n) = j(t_{\mathscr{A}}(a_1,\ldots,a_n)) \leqslant j(t'_{\mathscr{A}}(a_1,\ldots,a_n)) = t'_{\mathscr{A}_j}(a_1,\ldots,a_n)$$

for every $a_1,\ldots,a_n \in A_j$, so $\mathscr{A}_j$ also satisfies the inequality $t \leqslant t'$. □

Let us stop here for a moment to give some examples of varieties of sup-algebras.

Example 18. Let $\mathscr{Q} = (Q, *, 1_Q, \leqslant_Q)$ be a unital quantale and let $\top_Q$ be its top element. The *variety of left $\mathscr{Q}$-modules* is given by the type $\Omega = \Omega_1 = \{a\cdot \mid a \in Q\}$ and the set of identities

$$E = \{a\cdot(b\cdot x) = (a*b)\cdot x \mid a, b \in Q\}.$$

The *variety of topped left $\mathscr{Q}$-modules* (see Definition 2.4.2 of [6]) is given by the type $\Omega \cup \{\top\}$ and the set of identities

$$E \cup \{\top = \top_Q \cdot \top\}.$$

The *variety of unital left $\mathscr{Q}$-modules* is given by the type Ω and the set of identities

$$E \cup \{x = 1_Q \cdot x\}.$$

The *variety of pre-unital left $\mathcal{Q}$-modules* is given by the type Ω and the set of identities

$$E \cup \{x \leqslant 1_Q \cdot x\}.$$

The other way of forming quotient sup-algebras uses the notion of congruence. By a *sup-algebra congruence* we mean an equivalence relation ρ that is compatible with all operations and joins. The last means that if $a_i \, \rho \, b_i$ for all $i \in I$, then $(\bigvee_{i\in I} a_i) \; \rho \; (\bigvee_{i\in I} b_i)$. We denote the set of all congruences on $\mathscr{A}$ by $\mathsf{Con}(\mathscr{A})$.

It can be shown that kernels of sup-algebra homomorphisms are sup-algebra congruences. To prove a similar result for nuclei, we need the following lemma.

Lemma 19. *If j is a nucleus on a sup-algebra $\mathscr{A}$, then for all $a_i \in A_i,\ i \in I$,*

$$j\left(\bigvee_{i\in I} j(a_i)\right) = j\left(\bigvee_{i\in I} a_i\right).$$

Proof. The inequality $j(\bigvee_{i\in I} a_i) \leqslant j(\bigvee_{i\in I} j(a_i))$ follows because j is increasing and monotone. Conversely, since $j(a_i) \leqslant j(\bigvee_{i\in I} a_i)$ for each $i \in I$, we have $\bigvee_{i\in I} j(a_i) \leqslant j(\bigvee_{i\in I} a_i)$. Therefore, $j\,(\bigvee_{i\in I} j(a_i)) \leqslant j\,(j\,(\bigvee_{i\in I} a_i)) = j\,(\bigvee_{i\in I} a_i)$. □

Lemma 20. *Kernel of a nucleus $j : \mathscr{A} \to \mathscr{A}$ is a congruence on a sup-algebra $\mathscr{A}$.*

Proof. By Lemma 12, $\ker j$ is compatible with operations and by Lemma 19 it is compatible with joins. □

Let ρ be a congruence on a sup-algebra $\mathscr{A}$. Then we can define joins on the quotient set $A/\rho = \{[a] \mid a \in A\}$, where $[a] = \{b \in A \mid a\rho b\}$, by

$$\bigvee_{i\in I} [a_i] := \left[\bigvee_{i\in I} a_i\right], \tag{4.1}$$

and operations on A/ρ are defined in the usual manner. In this way we obtain a sup-algebra, which we denote by $\mathscr{A}/\rho$. From (4.1) we conclude that the order on the quotient sup-algebra $\mathscr{A}/\rho$ is given by

$$[a] \preceq [b] \Longleftrightarrow [a] \vee [b] = [b] \Longleftrightarrow (a \vee b)\, \rho\, b.$$

In the theory of ordered algebras, the order on the quotient is defined by

$$[a] \sqsubseteq [b] \Longleftrightarrow a \underset{\rho}{\leqslant} b,$$

where $a \underset{\rho}{\leqslant} b$ means that

$$a \leqslant a_1 \, \rho \, a_2 \leqslant a_3 \, \rho \, a_4 \leqslant \ldots \leqslant a_{n-1} \, \rho \, a_n \leqslant b$$

for some $a_1, \ldots, a_n \in A$ (see [2]). The next lemma shows that these two orders actually coincide.

Lemma 21. *Let $(L, \vee, \wedge)$ be a lattice and let ρ be an upper semilattice congruence on L. Then, for every $a, b \in L$,*

$$a \underset{\rho}{\leqslant} b \Longleftrightarrow (a \vee b)\, \rho\, b.$$

Proof. **Necessity.** Suppose that $a \underset{\rho}{\leqslant} b$, that is,

$$a \leqslant a_n \, \rho \, a_{n-1} \leqslant \ldots \leqslant a_6 \, \rho \, a_5 \leqslant a_4 \, \rho \, a_3 \leqslant a_2 \, \rho \, a_1 \leqslant b$$

for some $a_1, \ldots, a_n \in L$. Now $a_1 \, \rho \, a_2$ implies $(a_1 \vee b)\, \rho\, (a_2 \vee b)$, hence $b\, \rho\, (a_2 \vee b)$. A similar argument gives that $a_2\, \rho\, (a_4 \vee a_2)$. Now $b\, \rho\, (a_2 \vee b)$ and $a_2\, \rho\, (a_4 \vee a_2)$ imply that $(b \vee a_2)\, \rho\, (a_2 \vee a_4 \vee b)$, and hence

$b\,\rho\,(a_4 \vee b)$. Continuing in this manner we arrive at $b\,\rho\,(a_n \vee b)$. Taking upper bounds with a we obtain $(a \vee b)\,\rho\,(a \vee a_n \vee b)$, thus $(a \vee b)\,\rho\,(a_n \vee b)\,\rho\,b$ and $(a \vee b)\,\rho\,b$.

Sufficiency. If $(a \vee b)\,\rho\,b$, then we have a sequence $a \leqslant (a \vee b)\,\rho\,b$ between a and b. $\square$

A binary relation ρ on an ordered algebra $\mathscr{A}$ is said to satisfy the *closed chains condition* if $a \underset{\rho}{\leqslant} b \underset{\rho}{\leqslant} a$ implies $a\,\rho\,b$ for all $a, b \in A$. An *order-congruence* on an ordered algebra $\mathscr{A}$ is a congruence of the underlying algebra that satisfies the closed chains condition (see [2]).

Corollary 22. *Every lattice congruence on a lattice satisfies the closed chains condition.*

Proof. If $a \underset{\rho}{\leqslant} b \underset{\rho}{\leqslant} a$, then $(a \vee b)\,\rho\,b$ and $(a \vee b)\,\rho\,a$. Hence $a\,\rho\,b$ by transitivity. $\square$

From Corollary 22 we can conclude that the congruences on a sup-algebra $\mathscr{A}$ are precisely the order-congruences of $\mathscr{A}$ (considered as an ordered algebra) that are compatible with joins.

If $\mathscr{A}$ is a sup-algebra and $\rho \in \mathsf{Con}(\mathscr{A})$, then the natural surjection $A \to A/\rho$, $a \mapsto [a]$ will be denoted by $\rho^\natural$, and its right adjoint by $\rho^\natural_*$. By Lemma 11, the mapping

$$j_\rho = \rho^\natural_* \rho^\natural : A \to A$$

is a nucleus on $\mathscr{A}$.

Lemma 23 ([6], Proposition 2.2.11)**.** *If $\mathscr{A}$ is a sup-algebra and $\rho \in \mathsf{Con}(\mathscr{A})$, then* $\ker \rho^\natural = \ker j_\rho$.

Our main result in this section is the following.

Theorem 24. *Let $\mathscr{A}$ be a sup-algebra. Then there exists an isomorphism $\psi : \mathsf{Nuc}(\mathscr{A}) \to \mathsf{Con}(\mathscr{A})$ of posets. Moreover, for each $j \in \mathsf{Nuc}(\mathscr{A})$, $A_j \cong A/\psi(j)$ as sup-algebras.*

Proof. We define a mapping $\psi : \mathsf{Nuc}(\mathscr{A}) \longrightarrow \mathsf{Con}(\mathscr{A})$ by

$$\psi(j) := \ker j,$$

where $j \in \mathsf{Nuc}(\mathscr{A})$. By Lemma 20, $\ker j$ is indeed a sup-algebra congruence. By Proposition 2.2.8(5) of [6], ψ is an order-embedding. Thus, to show that ψ is an isomorphism of posets, it remains to prove its surjectivity. If $\rho \in \mathsf{Con}(\mathscr{A})$, then we can consider the natural surjection $\rho^\natural : A \to A/\rho$. By Lemma 23,

$$\psi(j_\rho) = \ker j_\rho = \ker \rho^\natural = \rho,$$

so ψ is surjective. Thus, we have established the isomorphism of posets $\mathsf{Con}(\mathscr{A})$ and $\mathsf{Nuc}(\mathscr{A})$.

Further, we take a nucleus j and define mappings $f : A/\ker j \to A_j$ and $g : A_j \to A/\ker j$ by

$$\begin{aligned} f([a]) &:= j(a), \\ g(a) &:= [a]. \end{aligned}$$

Obviously, f is well defined. The mapping f is a homomorphism, because using Lemma 12 we have

$$\begin{aligned} f(\omega_{A/\ker j}([a_1], \dots, [a_n])) &= f([\omega_A(a_1, \dots, a_n)]) \\ &= j(\omega_A(a_1, \dots, a_n)) \\ &= j(\omega_A(j(a_1), \dots, j(a_n))) \\ &= \omega_{A_j}(j(a_1), \dots, j(a_n)) \\ &= \omega_{A_j}(f([a_1]), \dots, f([a_n])) \end{aligned}$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega_n$,

$$f(\omega_{A/\ker j}) = f([\omega_A]) = j(\omega_A) = \omega_{A_j}$$

for every $\omega \in \Omega_0$, and, by Lemma 19,

$$\begin{aligned} f\left(\bigvee_{i\in I}[a_i]\right) &= f\left(\left[\bigvee_{i\in I} a_i\right]\right) = j\left(\bigvee_{i\in I} a_i\right) = j\left(\bigvee_{i\in I} j(a_i)\right) \\ &= \bigvee_{i\in I}^{j} j(a_i) = \bigvee_{i\in I}^{j} f([a_i]). \end{aligned}$$

For g we have

$$\begin{aligned} \omega_{A/\ker j}(g(a_1),\dots,g(a_n)) &= \omega_{A/\ker j}([a_1],\dots,[a_n]) \\ &= [\omega_A(a_1,\dots,a_n)] \\ &= [j(\omega_A(a_1,\dots,a_n))] \\ &= g(\omega_{A_j}(a_1,\dots,a_n)) \end{aligned}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $g(\omega_{A_j}) = g(j(\omega_A)) = [j(\omega_A)] = [\omega_A]$ for every $\omega \in \Omega_0$. It preserves joins because

$$g\left(\bigvee_{i\in I}^{j} a_i\right) = g\left(j\left(\bigvee_{i\in I} a_i\right)\right) = \left[j\left(\bigvee_{i\in I} a_i\right)\right] = \left[\bigvee_{i\in I} a_i\right] = \bigvee_{i\in I}[a_i] = \bigvee_{i\in I} g(a_i).$$

Hence g is a homomorphism.

For every $a \in A_j$, $f(g(a)) = f([a]) = j(a) = a$, so $fg = id$. Also $g(f([a])) = g(j(a)) = [j(a)] = [a]$ for every $a \in A$, thus $gf = id$. This completes the proof. ☐

5. A REPRESENTATION THEOREM

Let $\mathscr{A}$ be an ordered Ω-algebra and $\mathscr{P}(A)$ the power set of A. Define

$$\omega_{\mathscr{P}(A)}(D_1,\dots,D_n) := \{\omega_A(d_1,\dots,d_n) \mid d_i \in D_i,\ i = 1,\dots,n\}$$

for $n \in \mathbb{N}$, $\omega \in \Omega_n$, $D_i \in \mathscr{P}(A)$, $i = 1,\dots,n$, and

$$\omega_{\mathscr{P}(A)} := \omega_A\!\downarrow$$

if $\omega \in \Omega_0$. It is easy to see that $\mathscr{P}(\mathscr{A}) = (\mathscr{P}(A), \{\omega_{\mathscr{P}(A)} \mid \omega \in \Omega\}, \subseteq)$ is a sup-algebra.

Next we present a representation theorem for sup-algebras in terms of nuclei and quotients. This generalizes Theorem 3.1.2 in [7] and the main theorem of [9].

Theorem 25 (Representation Theorem). *If $\mathscr{A}$ is a sup-algebra, then there is a nucleus $j : \mathscr{P}(A) \to \mathscr{P}(A)$ such that $\mathscr{A} \cong \mathscr{P}(\mathscr{A})_j$ as sup-algebras.*

Proof. We define a mapping $j : \mathscr{P}(A) \to \mathscr{P}(A)$ by

$$j(D) = \left(\bigvee D\right)\downarrow.$$

It is routine to check that j is a closure operator, we show that j is a subhomomorphism. For $\omega_A(d_1,\dots,d_n) \in \omega_{\mathscr{P}(A)}(j(D_1),\dots,j(D_n))$, where $d_i \in j(D_i) = (\bigvee D_i)\downarrow$, $D_i \subseteq A$, $i = 1,\dots,n$, one has $d_i \leqslant \bigvee D_i$. Hence

$$\begin{aligned} \omega_A(d_1,\dots,d_n) &\leqslant \omega_A\left(\bigvee D_1,\dots,\bigvee D_n\right) \\ &= \bigvee\{\omega_A(a_1,\dots,a_n) \mid a_i \in D_i,\ i = 1,\dots,n\} \\ &= \bigvee \omega_{\mathscr{P}(A)}(D_1,\dots,D_n). \end{aligned}$$

Together with

$$\begin{aligned} j(\omega_{\mathscr{P}(A)}(D_1,\dots,D_n)) &= \Big(\bigvee \omega_{\mathscr{P}(A)}(D_1,\dots,D_n)\Big)\downarrow \\ &= \Big\{x\in A \mid x\leqslant \bigvee \omega_{\mathscr{P}(A)}(D_1,\dots,D_n)\Big\} \end{aligned}$$

we obtain that $\omega_{\mathscr{P}(A)}(j(D_1),\dots,j(D_n))\subseteq j(\omega_{\mathscr{P}(A)}(D_1,\dots,D_n))$ in $\mathscr{P}(A)$. If $\omega\in\Omega_0$, then clearly $\omega_{\mathscr{P}(A)}\leqslant j(\omega_{\mathscr{P}(A)})$.

One can easily see that $j(D)=D,\ D\subseteq A$ if and only if $D=d\downarrow$ for some $d\in A$. So

$$\mathscr{P}(A)_j=\{D\in\mathscr{P}(A)\mid D=j(D)\}=\{D\subseteq A\mid D=d\downarrow \text{ for some } d\in A\}.$$

Now define $\phi: A\to\mathscr{P}(A)_j$ by

$$\phi(a)=a\downarrow$$

for all $a\in A$. Then obviously ϕ is surjective, and $a\leqslant a'$ in A if and only if $a\downarrow\subseteq a'\downarrow$ in $\mathscr{P}(A)_j$. It remains to show that ϕ preserves operations. Take $n\in\mathbb{N}$, $\omega\in\Omega_n$ and $a_1,\dots,a_n\in A$. Note that

$$\begin{aligned} \omega_{\mathscr{P}(A)_j}(\phi(a_1),\dots,\phi(a_n)) &= \omega_{\mathscr{P}(A)_j}(a_1\downarrow,\dots,a_n\downarrow) \\ &= j(\omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)) \\ &= \Big(\bigvee \omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)\Big)\downarrow \\ &= \Big\{x\in A\mid x\leqslant \bigvee \omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)\Big\}. \end{aligned}$$

If $x\in\phi(\omega_A(a_1,\dots,a_n))$, then $x\leqslant\omega_A(a_1,\dots,a_n)\in\omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)$, and hence $x\leqslant\bigvee\omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)$. Conversely, let $x\in\omega_{\mathscr{P}(A)_j}(\phi(a_1),\dots,\phi(a_n))$. Since $\omega_A(a_1,\dots,a_n)$ is an upper bound of $\omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)$, we have

$$x\leqslant\bigvee\omega_{\mathscr{P}(A)}(a_1\downarrow,\dots,a_n\downarrow)\leqslant\omega_A(a_1,\dots,a_n)$$

and $x\in\omega_A(a_1,\dots,a_n)\downarrow=\phi(\omega_A(a_1,\dots,a_n))$. Consequently, $\phi(\omega_A(a_1,\dots,a_n))=\omega_{\mathscr{P}(A)_j}(\phi(a_1),\dots,\phi(a_n))$.

Also, if $\omega\in\Omega_0$, then

$$\phi(\omega_A)=\omega_A\downarrow=\Big(\bigvee(\omega_A\downarrow)\Big)\downarrow=j(\omega_A\downarrow)=j(\omega_{\mathscr{P}(A)})=\omega_{\mathscr{P}(A)_j}. \qquad\square$$

6. SUBALGEBRAS OF SUP-ALGEBRAS

In this section we study subalgebras and conuclei of sup-algebras. As in classical cases, they turn out to be in one-to-one correspondence.

Definition 26. *If $\mathscr{A}$ is a sup-algebra, then a subset $M\subseteq A$ is called a* subalgebra *of $\mathscr{A}$ if M is closed under operations and joins.*

Let $\mathscr{A}$ be a sup-algebra. We denote by $\mathsf{Sub}(\mathscr{A})$ ($\mathsf{Sub}_{mc}(\mathscr{A})$) the set of all subalgebras (resp. meet-closed subalgebras) of $\mathscr{A}$. Since intersections of subalgebras are subalgebras, we have the following result.

Proposition 27. *Let $\mathscr{A}$ be a sup-algebra. Then $\mathsf{Sub}(\mathscr{A})$ is a complete lattice.*

Definition 28. *Let $\mathscr{A}$ be a sup-algebra. A coclosure operator g on $\mathscr{A}$ is called a* conucleus *if it is a subhomomorphism.*

We denote by $\mathsf{Conuc}(\mathscr{A})$ ($\mathsf{Conuc}_{mp}(\mathscr{A})$) the set of all conuclei (resp. meet-preserving conuclei) on $\mathscr{A}$. These two sets are posets with respect to pointwise order.

Lemma 29. *If g is a conucleus on a sup-algebra $\mathscr{A}$, then $A_g = \{a \in A \mid g(a) = a\}$ is a subalgebra of $\mathscr{A}$.*

Proof. Let g be a conucleus on $\mathscr{A}$. As in the case of quantales (see [7], Theorem 3.1.3) one can show that A_g is closed under arbitrary joins. Furthermore, if $n \in \mathbb{N}$, $\omega \in \Omega_n, a_1, \ldots, a_n \in A_g$, then

$$\omega_A(a_1, \ldots, a_n) = \omega_A(g(a_1), \ldots, g(a_n)) \leqslant g(\omega_A(a_1, \ldots, a_n)).$$

Since g is decreasing, we conclude that $\omega_A(a_1, \ldots, a_n) = g(\omega_A(a_1, \ldots, a_n))$. □

Lemma 30. *If $\mathscr{A}$ is a complete lattice, $M \subseteq A$ is closed under joins and meets, and $a_i \in A$, $i \in I$, then*

$$\bigvee \left(M \cap \left(\bigwedge_{i \in I} a_i \right) \downarrow \right) = \bigwedge_{i \in I} \left(\bigvee (M \cap a_i \downarrow) \right).$$

Proof. Denoting $U := M \cap (\bigwedge_{i \in I} a_i)\downarrow$, $u := \bigvee U$, $v_i := \bigvee(M \cap a_i \downarrow)$ and $v := \bigwedge_{i \in I} v_i$ we need to prove that $u = v$.

To prove that $u \leqslant v$ it suffices to show that $u \leqslant v_i$ for each $i \in I$. If $x \in U$, then $x \in M$ and $x \leqslant a_i$ for every $i \in I$. Hence $x \leqslant v_i$ for every $i \in I$ and $x \leqslant \bigwedge_{i \in I} v_i = v$. Since v is an upper bound of U, we have $u \leqslant v$.

Conversely, since M is closed under joins and meets, $v_i \in M$ and $v = \bigwedge_{i \in I} v_i \in M$. Also $a_i \downarrow$ is closed under joins, so $v_i \leqslant a_i$ for each $i \in I$. Thus $v = \bigwedge_{i \in I} v_i \leqslant \bigwedge_{i \in I} a_i$, and $v \in U$. Consequently, $v \leqslant u$. □

Theorem 31. *Let $\mathscr{A}$ be a sup-algebra. Then there exists an isomorphism $\varphi : \mathsf{Sub}(\mathscr{A}) \longrightarrow \mathsf{Conuc}(\mathscr{A})$ of posets such that $M = A_{\varphi(M)}$ for each $M \in \mathsf{Sub}(\mathscr{A})$. Moreover, φ induces an isomorphism between posets $\mathsf{Sub}_{mc}(\mathscr{A})$ and $\mathsf{Conuc}_{mp}(\mathscr{A})$.*

Proof. We define mappings $\psi : \mathsf{Conuc}(\mathscr{A}) \longrightarrow \mathsf{Sub}(\mathscr{A})$ and $\varphi : \mathsf{Sub}(\mathscr{A}) \longrightarrow \mathsf{Conuc}(\mathscr{A})$ by

$$\begin{array}{rcl} \psi(g) & := & A_g, \\ \varphi(M) & := & g_M, \end{array}$$

where

$$g_M(a) = \bigvee \{m \in M \mid m \leqslant a\} = \bigvee (M \cap a \downarrow)$$

for any $a \in A$. It is not difficult to see that g is a coclosure operator. However, we also have that for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$,

$$\begin{aligned} & \omega_A(g(a_1), \ldots, g(a_n)) \\ & = \omega_A \left(\bigvee \{m \in M \mid m \leqslant a_1\}, \ldots, \bigvee \{m \in M \mid m \leqslant a_n\} \right) \\ & = \bigvee \{\omega_A(m_1, \ldots, m_n) \mid m_i \in M, m_i \leqslant a_i,\ i = 1, \ldots, n\} \\ & \leqslant \bigvee \{m \in M \mid m \leqslant \omega_A(a_1, \ldots, a_n)\} \\ & = g(\omega_A(a_1, \ldots, a_n)). \end{aligned}$$

If $\omega \in \Omega_0$, then $\omega_A \in M \cap \omega_A \downarrow$ because M is a subalgebra. Thus $\omega_A \leqslant g_M(\omega_A)$. We have shown that g is a subhomomorphism and hence a conucleus.

To prove that φ is monotone we suppose that $M \subseteq N$, $M, N \in \mathsf{Sub}(\mathscr{A})$. Take $a \in A$. Then $M \cap a\downarrow \subseteq N \cap a\downarrow$, whence $g_M(a) = \bigvee(M \cap a \downarrow) \leqslant \bigvee(N \cap a \downarrow) = g_N(a)$. Thus $g_M \leqslant g_N$.

Let us show that ψ is monotone. Suppose that $g \leqslant h$, $g, h \in \mathsf{Conuc}(\mathscr{A})$. If $a \in A_g$, then $a = g(a) \leqslant h(a)$. Since h is a conucleus, $h(a) \leqslant a$. Therefore $a = h(a)$ and $a \in A_h$. This proves the inclusion $A_g \subseteq A_h$.

To prove that $\psi\varphi = id$ we have to show that $A_{g_M} = M$ for every $M \in \mathsf{Sub}(\mathscr{A})$. Take $x \in M$. Then $g_M(x) = \bigvee(M \cap x\downarrow) = x$, and so $x \in A_{g_M}$. Conversely, let $a \in A_{g_M}$, i.e. $a = \bigvee(M \cap a\downarrow)$. Since M is closed under joins, $a \in M$, and therefore $A_{g_M} \subseteq M$.

To verify the equality $\varphi\psi = id$ we need to show that $g_{A_g} = g$ for every $g \in \mathsf{Conuc}(\mathscr{A})$. Take $a \in A$. Then $g_{A_g}(a) = \bigvee(A_g \cap a\downarrow)$. Since $g(a) \in A_g \cap a\downarrow$, we have $g(a) \leqslant \bigvee(A_g \cap a\downarrow) = g_{A_g}(a)$. On the other hand, if $x \in A_g \cap a\downarrow$, then $x \leqslant a$ and $x = g(x) \leqslant g(a)$. So $g(a)$ is an upper bound of $A_g \cap a\downarrow$ and $g_{A_g}(a) \leqslant g(a)$. Consequently, $g_{A_g} = g$. Thus we have established an isomorphism $\mathsf{Sub}(\mathscr{A}) \cong \mathsf{Conuc}(\mathscr{A})$.

If $M \in \mathsf{Sub}_{mc}(\mathscr{A})$, then $\varphi(M) = g_M$ preserves meets because of Lemma 30. Also, if $g \in \mathsf{Conuc}_{mp}(\mathscr{A})$, then $\psi(g) = A_g$ is closed under meets, because

$$g\left(\bigwedge_{i\in I} a_i\right) = \bigwedge_{i\in I} g(a_i) = \bigwedge_{i\in I} a_i$$

for $a_i \in A_g$, $i \in I$. □

The following result can be proved exactly as Proposition 32 of [10].

Proposition 32. *For every sup-algebra $\mathscr{A}$, the posets $\mathsf{Sub}_{mc}(\mathscr{A})$ and $\mathsf{Nuc}(\mathscr{A})$ are dually isomorphic.*

For a poset X we denote its dual poset by X^d. We can summarize our main results in the following form.

Corollary 33. *For a sup-algebra $\mathscr{A}$ we have the following isomorphisms and order-embeddings of posets*:

$$\begin{array}{ccccccc} \mathsf{Con}(\mathscr{A})^d & \cong & \mathsf{Nuc}(\mathscr{A})^d & \cong & \mathsf{Sub}_{mc}(\mathscr{A}) & \subseteq & \mathsf{Sub}(\mathscr{A}) \\ & & & & \cong & & \cong \\ & & & & \mathsf{Conuc}_{mp}(\mathscr{A}) & \subseteq & \mathsf{Conuc}(\mathscr{A}). \end{array}$$

Remark 34. Since $\mathsf{Nuc}(\mathscr{A})$ and $\mathsf{Sub}(\mathscr{A})$ are complete lattices (see Proposition 13 and Proposition 27), all posets in the above corollary are actually complete lattices and all isomorphisms are isomorphisms of complete lattices. Moreover, $\mathsf{Con}(\mathscr{A})$ is a quantale with respect to inclusion and relational product of congruences. Therefore one can also introduce a quantale structure on $\mathsf{Nuc}(\mathscr{A})$, $\mathsf{Sub}_{mc}(\mathscr{A})^d$, and $\mathsf{Conuc}_{mp}(\mathscr{A})^d$.

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Sup-algebrate faktor- ja alamalgebrad

Xia Zhang ja Valdis Laan

Järjestatud algebrat nimetatakse sup-algebraks, kui see järjestatud hulk, millel ta on defineeritud, on täielik võre ja tema tehted on iga argumendi suhtes kooskõlas ülemiste rajadega. Selles artiklis on uuritud sup-algebrate faktor- ja alamalgebraid. On näidatud, et sup-algebra kongruentside võre on isomorfne tema tuumade võrega ja duaalselt isomorfne tema alumise raja suhtes kinniste alamalgebrate võrega. Samuti on tõestatud, et sup-algebra alamalgebrate võre on isomorfne tema kotuumade võrega.